LOCAL Lp-INVERSE THEOREMS OF A GENERAL ORDER FOR LINEAR COMBINATIONS OF CERTAIN LINEAR POSITIVE OPERATORS
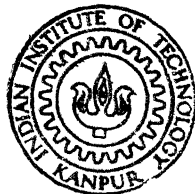
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In Partial Fulfilment of the Requirements
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DEDICATED

TO

MY PARENTS

CERTIFICATE

Certified that the work presented in this thesis entitled 'LOCAL L_p-INVERSE THEOREMS OF A GENERAL ORDER FOR LINEAR COMBINATIONS OF   CERTAIN LINEAR POSITIVE OPERATORS' by Shree Peri Sarweswara Avadhani has been carried out under my supervision and that this has not been submitted elsewhere for a degree or diploma.

December, 1982

(R.K.S. RATHORE)

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CONTENTS

SYNOPSIS

Local inverse theorems in the approximation by linear
operators have been of considerable interest during the
past decade. The initial problems considered in this
connection concerned approximation by sequences of linear
positive operators. Subsequently, their linear combinations,
which could provide a faster degree of approximation, were
taken up. Local inverse theorems for several sequences of
linear positive operators and their linear combination:
have been obtained during this period. The orders of
approximation studied in these local inverse theorems have
been $n^{-\alpha}$, α a positive number.

A representative inverse result which has been shown
to be valid for several sequences of linear positive operators
is: $||L_n(f;t) - f(t)||_{L_p[a,b]} = O(n^{-\alpha/2})$ implies that
$\omega_2(f,h,p,[a_1,b_1]) = O(h^\alpha)$, where $a < a_1 < b_1 < b$. A similar
result for linear combinations $L_n(\cdot,k,t)$ of the operators
L_n defined by

$$L_n(\cdot,k,t) = \sum_{j=0}^{k} \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i} L_{d_j n}(\cdot,t),$$

shown to be valid for several sequences $\{L_n\}$ of linear
positive operators is: $||L_n(f,k,t) - f(t)||_{L_p[a,b]} = O(n^{-\alpha/2})$
implies that $\omega_{2k+2}(f,h,p,[a_1,b_1]) = O(h^\alpha)$, where
$a < a_1 < b_1 < b$.

It may also be noted that the classical Bernstein inverse theorems too concern with the order $O(n^{-\alpha})$ of approximation by trigonometric polynomials of best approximation, for instance.

The present thesis considers local L_p-inverse theorems for linear combinations of linear positive operators of a much more general order $\varphi(n^{-1/2})$, where the order function φ is subject only to the following constraints: φ is a positive function on $(o,c]$, for some $c > 0$, such that /

1.
$$\sup_{t \,\varepsilon\, (0,c]} \frac{\varphi(t)}{\varphi(th)} \leq K_\varphi(h) \,,$$

where $K_\varphi(h)$ is a positive function defined for $h \,\varepsilon\, (0,1)$.

2. $\sup\limits_{h \geq \delta} K_\varphi(h) < \infty$, for all $\delta \,\varepsilon\, (0,1)$

and

3. $h^r K_\varphi(h) \to 0$ as $h \to 0$, for an appropriate positive integer r. Let Φ_r denote the collection of functions φ satisfying the above properties.

The function $\varphi(t) = t^\alpha (\alpha > 0)$, which corresponds to the order $n^{-\alpha}(= \varphi(n^{-1}))$ belongs to φ_r, for any integer $r > \alpha$.

Some non-trivial examples of $\varphi \,\varepsilon\, \Phi_r$ are as follows:

1. For all $t \,\varepsilon\, (0,c]$,

$$\varphi(t) = t^\alpha (\log^p(\tfrac{1}{t}))^q \,\varepsilon\, \Phi_r,$$

where p is a positive integer, $q > 0$, $0 < \alpha < r$, $c > 0$ with

$\log^p 1/c > 0$ and $\log^p t$ denotes the function $\log (\log...(\log t))$ (p times).

2. For all $t \in (0,c]$,

$$\varphi(t) = \frac{t^{\alpha}}{(\log^p(\frac{1}{t}))^q} \in \Phi_r,$$

where p is a positive integer and $q > 0$, $0 < \alpha < r$ and $c > 0$ such that $\log^p \frac{1}{c} > 0$.

3. Let $a < b$ be extended real numbers and $r > k \in \mathbb{N}$ and $f \in L_p[a,b]$ ($1 \le p < \infty$) be a function which is not a polynomial of degree $\le k-1$. Then

$$\varphi(t) = \omega_k(f,t,p,[a,b]) \in \Phi_r,$$

where $t \in (0,c]$, $c < \frac{b-a}{2k}$.

The representative $O(\varphi)$-inverse result which, in this thesis, has been shown to be valid for several sequences of linear combinations of linear positive operators reads:
Let $\varphi \in \Phi_{2k+2}$. Then

$$||L_n(f,k,t) - f(t)||_{L_p[a,b]} = O(\varphi(n^{-1/2}))$$

implies that

$$\omega_{2k+2}(f,t,p,[a_1,b_1]) = O(\varphi(t)),$$

where $a < a_1 < b_1 < b$.

The sequences and classes of operators which have been considered in the thesis are:

(a)   the operators U_n of Sikkema and Rathore defined for $f \varepsilon L_p(-\infty,\infty)$ by

$$U_n(f;x) = \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} f(t)\, \beta^n(x-t)\, dt,$$

where

$$\alpha(n) = \int_{-\infty}^{\infty} \beta^n(t)\, dt$$

and $\beta \varepsilon L_1(-\infty,\infty)$ is a bell-shaped function satisfying: $\beta(0) = 1$, β infinitely differentiable in a neighbourhood of origin, $\beta''(0) \neq 0$ and $\sup \{ \beta(t): |t| \geq \delta \} < 1$ for every $\delta > 0$.

(b)   The Bernstein-Kantorovitch polynomials P_n defined for $f \varepsilon L_p[0,1]$ by

$$P_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k}(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\, dt.$$

(c)   The generalized Müller's operators T_λ defined for $f \varepsilon L_p(o,\infty)$ as,

$$T_\lambda(f;x) = \frac{x^{\alpha-1}}{a(\lambda)} \int_{o}^{\infty} u^{-\alpha}\, G^\lambda(x/u)\, f(u)\, du,$$

where

$$\bar{a}(\lambda) = \int_{o}^{\infty} u^{\alpha-2}\, G^\lambda(u),$$

α is a real number, G is a measurable function differentiable in a neighbourhood of the point '1', $\sup \{G(u): |u-1| \geq \delta\} < G(1)$ for all $\delta > 0$ and $(u^{-a}+u^{b})\, G(u)$ being bounded on (o,∞) for some $a,b > 0.$

(d) The regular exponential type operators S_n defined for $f \in L_p[A,B]$ (A,B distinct extended real numbers) by

$$S_n(f;t) = \int_A^B W(n,t,u)\, f(u)\, du,$$

where the kernel $W(n,t,u)$ satisfies:

(i) $\quad \int_A^B W(n,t,u)\, du = 1$

(ii) $\quad$ For $u \in (A,B)$, $\int_A^B W(n,t,u)\, dt = a(n) \to 1 \ (n \to \infty)$,

$a(n)$ being a rational function of n,

(iii) $\quad \dfrac{\partial W}{\partial t} = \dfrac{n}{p(t)} (u-t) W, \ u,t \in (A,B)$

so that

$$\frac{d^k}{dt^k} S_n(f;t) = \int_A^B \frac{\partial^k}{\partial t^k} W(n,t,u)\, f(u)\, du$$

and

(iv) $\quad t^m p(t)\, W(n,t,u) \to 0$ as $t \to A,B$

for all sufficiently large values of n, and

(e) the generalized Jackson operators

$$L_{n,p}(f;x) = \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin n \frac{(t-u)}{2}}{\sin \frac{t-u}{2}} \right)^{2p} f(t)\, dt$$

with $\quad A_{n,p} = \int_{-\pi}^{\pi} \left(\dfrac{\sin \frac{nt}{2}}{\sin t/2} \right)^{2p} dt$,

where p is a positive integer.

Apart from considering the linear combinations of the above operators we have also considered interpolatory modifications

$P_{n,m}$ and $S_{n,m}$ of the operators P_n and S_n, respectively, defined by

$$P_{n,m}(f;t) = (n+1) \sum_{\nu=0}^{n} \sum_{j=0}^{m} \frac{n^{j/2}}{j!} \; p_{n\nu}(t) \times$$

$$\times \int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta_{n^{-1/2}}^{j} f(u) du,$$

where $\prod_{i=0}^{j-1} (t-u-i/n^{1/2})$ for $j = 0$ is interpreted as 1, $f(u)$ is regarded as zero when $u > 1$ and

$$p_{n\nu}(t) = \binom{n}{\nu} t^{\nu} (1-t)^{n-\nu}$$

and

$$S_{n,m}(f;t) = \int_{A}^{B} \frac{W(n,t,u)}{1+|u-t|^{m_0+2}} \times$$

$$\times \{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta_{n^{-1/2}}^{j} f(u) \} du$$

where $m \leq m_0$ and $W(n,t,u)$ is the kernel of the operators S_n.

In addition to obtaining $0(\varphi)$-inverse theorems for the above operators, we have also introduced and obtained $o(\varphi)$-inverse theorems. A $o(\varphi)$-inverse result for linear combinations $L_n(\cdot,k,t)$ reads as:

Let $\varphi \varepsilon \Phi_{2k+2}$. Then

$$||L_n(f,k,t) - f(t)||_{L_p}[a,b] = o(\varphi(n^{-1/2}))$$

implies that

$$\omega_{2k+2}(f,h,p,[a_1,b_1]) = o(\varphi(h)),$$

where $a < a_1 < b_1 < b$.

This has been proved to be valid for linear combinations and interpolatory modifications defined above.

The main ingradients of the proofs of earlier known inverse theorems have been Bernstein type inequalities and moments of the operators, a lemma of Berens and Lorentz and Peetre's K-functionals or Steklov type means. For obtaining φ-inverse theorems in the thesis, two generalisations of the Berens-Lorentz lemma and two generalisations of the result :

$$\omega_{k+1}(f,t,p,[a,b]) = O(t^\alpha)$$

implies that

$$\omega_k(f,t,p,[a,b]) = O(t^\alpha),$$

whenever $\alpha < k$, have been obtained and are found to be crucial. The technique developed for φ-inverse theorems seems to be of a general applicability also with respect to sup-norm and several more sequences of linear positive operators.

The thesis is devided into six chapters. Chapter 0 is introductory in nature and contains some basic definitions, notations and a brief survey of the chapterwise contents of the thesis. In Chapter I we obtain φ-inverse theorems for the linear combinations $U_n(.,k,t)$. In Chapter II

φ-inverse theorems for linear combinations $P_n(\cdot, k, t)$ and interpolatory modifications $P_{n,m}(\cdot, t)$ have been obtained. Chapter III is a study of φ-inverse theorems of linear combinations of generalised Muller's operators T_λ. In Chapter IV φ-inverse theorems for linear combinations $S_n(\cdot, k, t)$ and interpolatory modifications $S_{n,m}(\cdot, t)$ of regular exponential type operators have been obtained. Finally, in Chapter V, φ-inverse theorems have been obtained for linear combinations $L_{n,p}(\cdot, k, t)$ of the generalised Jackson operators.

INTRODUCTION AND CONTENTS
OF THE THESIS

0.1 Introduction.

Ditzian and May [22] proved the following theorem :

Let $0 < a < a_1 < b_1 < b < 1$, $f \varepsilon L_p[0,1]$ and $||P_n f - f||_{L_p[a,b]}$

$\leq K n^{-\alpha/2}$ for some $0 < \alpha < 2$, then, for all $h \leq \frac{1}{2}$ min $(b-b_1, a_1-a)$,

$$\omega_2(f,h,p,[a_1,b_1]) = \sup_{|\tau|<h} ||\Delta_\tau^2 f||_{L_p[a_1,b_1]} \leq Mh^\alpha, \quad \text{where}$$

$$P_n(f,t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} (n+1) \int_{k/n+1}^{(k+1)/n+1} f(u) \, du,$$

is the nth Bernstein-Kantorovitch polynomial of f. Sinha [61] ,
subsequently, generalised this result to the linear combinations

$$P_n(f,k,t) = \sum_{j=0}^{k} \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j-d_i} P_{d_j n}(f;t)$$

of the Bernstein-Kantorovitch polynomials as follows : Let
$0 < a < a_1 < b_1 < b < 1$, $f \varepsilon L_p[0,1]$ and
$||P_n(f,k,t) - f(t)||_{L_p[a,b]} = O(n^{-\alpha/2})$ for $0 < \alpha < 2k+2$, then

$$\omega_{2k+2}(f,h,p,[a_1,b_1]) = \sup_{|\tau|<h} ||\Delta_\tau^{2k+2} f||_{L_p[a_1,b_1]} = O(h^\alpha).$$

The work of the present thesis arose out of the following
question to which we addressed ourselves to begin with :
Suppose $||P_n(f,k,t) - f(t)||_{L_p[a,b]}$ or $||P_n(f;t)-f(t)||_{L_p[a,b]}$
is of $O(n^{-\alpha/2} \log n)$ or $||P_n(f,k,t)-f(t)||_{L_p[a,b]}$ or

$||P_n(f;t)-f(t)||_{L_p}[a,b]$ is of $O(n^{-\alpha/2}/\log n)$, then what can we say about the smoothness of the function f? The answer to this question turned out to be : $\omega_{2k+2}(f,t,p,[a_1,b_1]) = O(t^{\alpha} \log (1/t))$, $\omega_2(f,t,p, [a_1,b_1]) = O(t^{\alpha} \log (1/t))$, $\omega_{2k+2}(f,t,p, [a_1,b_1]) = O(t^{\alpha}/\log (1/t))$ and $\omega_2(f,t,p, [a_1,b_1]) = O(t^{\alpha}/\log (1/t))$, respectively. The form of the above answer was quite suggestive and it led us to a consideration of the more general problem of obtaining inverse theorems of a general order. The present thesis is the result of this undertaking.

The main ingredients of the proofs of earlier theorems were Bernstein type inequalities and moments of the operators, a lemma of Berens-Lorentz and Peetre's K-functionals or Steklov type means. We found that two generalisations of Berens-Lorentz lemma and two generalisations of the result : $\omega_{k+1}(f,t,p, [a,b]) = O(t^{\alpha})$ $(t \to 0)$ implies that $\omega_k(f,t,p,[a,b]) = O(t^{\alpha})$, whenever $\alpha < k$, turn out to be sufficient to provide us with a machinery by which the problem of a general order inverse theorem could be solved. In the present thesis, we have illustrated this for five well known classes or sequences of operators. However, it may be easily seen that the technic is of quite a general applicability (also with respect to sup-norm case). We have restricted ourselves only to the local approximation theorems. This is because we feel that approximation by linear operators seems specially tailored for taking advantage of local smoothness of

functions, as may be concluded from discussion of the subsequent section.

0.2 Local approximation.

Consider $f(x) = |x - \frac{1}{2}|^{1/10}$, $x \in [0,1]$. The polynomial P_n of best approximation of degree n to this function on $[0,1]$ in sup-norm, has precise order $n^{-1/10}$ of approximation. In view of the Chebyshev equi-oscillation property it may be concluded that the same order of approximation persists even on an interval $[.5+\delta,1]$ for any fixed $\delta \in (0,.5)$ (Infact, in the present case, sup $\{|f(x)-P_n(x)| : x \in [0,1]\} = |f(1)-P_n(1)|$). However, it may be noted that the function is badly behaved only in a neighbourhood of $x = 1/2$ and that on $[.5+\delta,1]$ it is a nice analytic function. Thus the polynomial of best approximation does not necessarily take advantage of local nice behaviour of functions. On the other hand, consider $B_n(f;t)$, the nth Bernstein polynomial of the same function. The order of approximation on $[0,1]$ by $B_n(f;t)$ is $O(n^{-\frac{1}{20}})$, which is poor in comparison with the order of approximation by P_n's. However, on $[.5+\delta,1]$ with δ as above, the order of approximation by B_n's is $O(n^{-1})$ and, therefore, B_n's indeed, take advantage of the local smoothness of functions in local approximation. Further, the linear combinations $\mathcal{L}_n^{[2k]}$ of B_n's defined by Butzer [14] approximate the function f on $[.5+\delta,1]$ with the order $n^{-(k+1)}$. This type of behaviour, i.e., providing a better approximation in the regions where the function is better behaved, is commonly

shared by several well known sequences of linear approximation operators. Some of the works concerned with the degree of (local/general) approximation by linear positive operators are those of Berens and Devore [9] , Bojanic and Shisha [13] , Hoeffding [26], Korovkin [30,31] , Lorentz [35,36], May [40], Müller [45] , Müller and Maier [46] , Rathore[48,50] and Wood [68] and others.

O.3 Linear combinations of linear positive operators.

Among linear approximation operators, linear positive operators occupy a special place of interest (Korovkin [31]). However, the positivity of such operators places a severe constraint on their degree of approximation (Devore [19] and Korovkin [31]). To compensate for this several methods, e.g., using their linear combinations (Butzer [14], Rathore [48,49] May [39] , Ditzian [20], Rathore and Agrawal [52] , Rathore and Singh [53] , Singh [60], Kunwar [32], Agrawal [1] , Sinha [61] and Winslin [67] etc.), the linear combinations of iterates of linear positive operators (Miccheli [42], Agrawal [1] , Bleimann, Junggeburth and Stark [12] , Butzer and Berens[16,p105]),multiplying the kernels of linear positive operators by suitable factors so as to produce kernels with appropriate finite oscillations (Stark [62,63] , Hoff [27]; see also Ditzian and Freud [23]) and interpolatory modifications recently considered by Sinha [61], etc. have been proposed in the literature. Most of the above methods have been studied

only with respect to the sup-norm. The study of linear combinations and interpolatory modifications of Sinha [61], however, is with respect to L_p-norms.

0.4 Order of approximation.

Approximation theorems concerned with the degree of approximation **are mainly** categorised as direct, inverse and saturation theorems. Indeed, a direct theorem tells the degree of approximation from a known smoothness property of a function, the inverse theorem infers the degree of smoothness from a given degree of approximation and a saturation theorem concerns with the fastest degree of approximation possible for a given method.

In this connection, most of the results particularly on inverse theorems for linear combinations etc. of linear positive operators, available in the literature, concern with the order $O(n^{-\alpha})$, α a real number, of approximation (Agrawal [1] , Ditzian [21], Kunwar [32] , May [39] , Rathore [48] Rathore and Singh [53], Sinha [61]and Winslin [67]). In the classical part of approximation theory (concerned with polynomials and entire functions of exponential type etc. of best approximation) also, the Bernstein theorems (which probably are the first known inverse theorems) correspond to the very order $O(n^{-\alpha})$. Even for basic sequences and classes of linear positive operators the inverse theorems available are with respect to the order $O(n^{-\alpha})$ of approximation (e.g. in the works

of Becker [2,3]   , Becker, Kucharski and Nessel [4]   , Becker
and Nessel [5 ,6 ,8] , Berens and Lorentz [10] , Butzer and
Berens [16] , Butzer and Nessel [17] , Devore [19] , Ditzian
[20], Ditzian and May [22] and de Leeuw [18]etc.'.

The basic theme of this thesis is to extend the degree
of approximation results (mainly, inverse theorems) to orders
of approximation more general than $O(n^{-\alpha})$. The function
which occurs in the general orders $O(\varphi(n^{-1/2}))$, with which we
concern ourselves in the thesis, is only subject to the
following restrictions :

$$\sup_{t\varepsilon(0,c]} \frac{\varphi(t)}{\varphi(th)} \leq K_\varphi(h), \text{ where for any } \delta > 0,$$

$$\sup_{1>h\geq\delta} K_\varphi(h) < \infty \qquad \text{and for an appropriate positive integer } r.$$

$h^r K_\varphi(h) \to 0$ as $h \to 0$. We may note that $\varphi(t) = t^\alpha$ (which
corresponds to approximation orders of type $n^{-\alpha}$) satisfies
the above conditions for any integer $r > \alpha$. In view of this,
the earlier known inverse theorems turn out to be very special
cases of the results obtained in the thesis. Some other
examples of functions φ with the above properties are

$$\varphi(t) = t^\alpha(\log^p (1/t))^q \qquad (\alpha < r),$$

$$\varphi(t) = t^\alpha/(\log^p 1/t)^q \qquad (\alpha < r)$$

and (a very general case)

$$\varphi(t) = \omega_k(t) \qquad (k < r),$$

where $\omega_k(t)$ is a kth modulus of continuity (with respect to

any of sup or integral L_p-norms) of some function on some interval.

In the thesis, in addition to $O(\varphi)$-inverse theorems, we also obtain inverse theorems of $o(\varphi)$-type. These have the flavour of the little-o Zygmund theorem as described in Natanson [47, Theorem 3, p. 109].

The approximation operators which we study with respect to the general order φ of approximation are linear combinations or interpolatory modifications of the following five sequences or classes of linear positive operators : (a) convolution operators U_n (Sikkema and Rathore [58]) on $(-\infty,\infty)$ generated by powers of bell-shaped functions as kernels, (b) Bernstein-Kantorovitch polynomials P_n (Kantorovitch [29]), (c) generalised Müller operators T_λ (Kunwar [32, 33]) which are also generated by powers of a fixed function as kernels on $(0,\infty)$, (d) Regular exponential type operators S_n (Sinha [61], Agrawal [1] and May [39]) and (e) generalised Jackson operators (Schurer [55,56] Rathore [50] and Lorentz [36]).

0.5   Some definitions and notations.

In this section we give some basic definitions and notations which we use throughout the thesis.

The symbols $\mathbb{R}, \mathbb{R}^+, \mathbb{N}$ and $\mathbb{N}^o$ denote the set of real numbers, positive real numbers, positive integers and non-negative integers, respectively. The proofs of various results in the

thesis extensively utilize constants which are independent of f
(the function under consideration) and n (in U_n, P_n, S_n and
$L_{n,p}$) or λ (in T_λ). Such constants will be denoted by
M, M_1, M_2, ..., M_1', ... etc. Thus, generally the presence of 'M'
would implicitly mean that the constant is independent of f
and n or λ .

Let $1 \leq p < \infty$. Then $L_p[a,b]$ is defined as the class of
all complex valued measurable functions f (with the usual
identification of functions equal a.e.) for which $\int_a^b |f(x)|^p dx < \infty$,
where the integration is in the Lebesgue sense and the norm
in $L_p[a,b]$ is defined by

$$||f||_{L_p[a,b]} = (\int_a^b |f(x)|^p dx)^{1/p} .$$

The space $L_\infty[a,b]$, similarly, consists of complex valued
measurable functions which are essentially bounded on (a,b)
and it is normed by

$$||f||_{L_\infty[a,b]} = \inf \{M : |f(x)| \leq M \text{ a.e. on } [a,b] \}.$$

The classes of all absolutely continuous functions over
$[a,b]$ and functions of bounded variation over $[a,b]$ will be
denoted, respectively, by A.C. $[a,b]$ and B.V. $[a,b]$. The
class of k times continuously differentiable functions on $\mathbb{R}$
which have a compact support is denoted by C_o^k.

Next, we present some general definitions which are
quite often used in the thesis. As per standard practice,
numbers in the square brackets are those of pertinent references
listed at the end of the thesis.

0.5.1 **Newton's forward differences** [25]. Let f be a complex valued function over $\mathbb{R}$ and $m \in \mathbb{N}$. Then, the mth forward difference of the function f at a point t, of step length δ, is defined as

$$\Delta_\delta^m f(t) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(t + j\delta).$$

Convetionally, we write $\Delta_\delta^0 f(t) = f(t)$.

0.5.2 **Newton's interpolation polynomials** [25]. If f is a complex valued function over $\mathbb{R}$ and $m \in \mathbb{N}$, then the Newton's interpolation polynomial $P_m(t)$ of degree m, which interpolates f at the equidistant points t_i, $i = 0,\ldots,m$, is given by

$$P_m(t) = \{ \sum_{j=0}^{m} \frac{1}{j!} \delta^j \prod_{i=0}^{j-1} (t-t_i) \Delta_\delta^j f(t_o) \},$$

where $\delta = t_{i+1} - t_i$, $i = 0,\ldots,m-1$ and $\prod_{i=0}^{j-1} (t-t_i) = 1$ if $j = 0$.

0.5.3 **Hardy-Littlewood majorant** [69]. For any $f \in L_p[a,b]$ $(1 < p < \infty)$, the Hardy-Littlewood majorant of f is defined as

$$H_f(x) = \sup_{\xi \neq x} \frac{1}{\xi - x} \int_x^\xi f(t) dt \quad (a \leq \xi \leq b).$$

0.5.4 **Integral modulus of smoothness** [65]. For any $f \in L_p[a,b]$ $(1 \leq p < \infty)$, the integral modulus of smoothness of f of order m is defined as

$$\omega_m(f,h,p,[a,b]) = \sup_{0 \leq \delta \leq h} ||\Delta_\delta^m f(t)||_{L_p[a,b-m\delta]}.$$

0.5.5 **Steklov means** [61] . Let $f \in L_p [a,b] (1 \leq p < \infty)$.

Then, for $\eta > 0$, the Steklov means $f_{\eta,m}$ of order m, corresponding

to f, is defined as follows :

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} (f(t) + (-1)^{m-1} \Delta_m^m \underset{\underset{i=1}{\Sigma t_i}}{} f(t)) dt_1 \ldots dt_m.$$

In the above integral we put $f(t) = 0$ if t does not belong to the

domain (a,b) of definition of f.

0.5.6 **Linear combinations** [48] . Given a sequence $\{L_n(\cdot,t)\}$

of linear positive operators, we define their linear

combinations $L_n(\cdot,k,t)$ as follows :

Let $d_0,\ldots,d_k$ be k+1 positive numbers such that $L_{d_j n}$'s

are meaningful. Then

$$L_n(\cdot,k,t) = \frac{1}{\Delta_k} \begin{vmatrix} L_{d_0 n}(\cdot,t) & d_0^{-1} & \cdots & d_0^{-k} \\ L_{d_1 n}(\cdot,t) & d_1^{-1} & \cdots & d_1^{-k} \\ \vdots & & & \\ L_{d_k n}(\cdot,t) & d_k^{-1} & \cdots & d_k^{-k} \end{vmatrix}$$

where Δ_k is the determinent obtained from the displayed

determinent after replacing the entries of the first column

by 1.

These combinations can also be written as [39]

$$L_n(\cdot,k,t) = \sum_{j=0}^{k} C(j,k) L_{d_j n}(\cdot,t)$$

where the coefficients C(j,k) are given by

$$C(j,k) = \begin{cases} \prod_{\substack{i=0 \\ i \neq j}}^{k} \dfrac{d_j}{d_j - d_i} & , \quad k \neq 0 \\[2em] 1 & , \quad k = 0, \end{cases}$$

and they satisfy

$$\sum_{j=0}^{k} C(j,k) \, d_j^{-m} = \delta_{mo} \; ,$$

for m = 1,2,...,k, where δ_{ij} is the Kronecker-delta equalling 1 if i = j and 0 otherwise. (Thus, the linear combination $L_n(.,k,t)$ is the kth degree Lagrange interpolation polynomial of $L_{x^{-1}n}(.,t)$ (being regarded as function of x) interpolating at

$x = d_0^{-1}, d_1^{-1}, ..., d_k^{-1}$ and evaluated at x = 0 [25].).

0.6 Certain basic results.

This section is a collection of some general results which will be used in the thesis.

We start with a lemma (Timan [65 , P.107]) which expresses an mth forward difference of a function in terms of an integral involving the mth derivative of the function.

Lemma 0.6.1. Let f ε $L_p[a,b]$ (1 ≤ p < ∞) and have an mth derivative such that

$f^{(m-1)}$ ε A.C.[a,b] and $f^{(m)}$ ε $L_p[a,b]$. Then

$$\Delta_\delta^m f(t) = \int_0^\delta \cdots \int_0^\delta f^{(m)}\left(t + \sum_{i=1}^m y_i\right) dy_1 \cdots dy_m, \quad t \in [a, b - m\delta].$$

The next lemma [64, 69] gives a bound for the Hardy Littlewood majorant of a function.

Lemma 0.6.2. Let $f \, \varepsilon \, L_p[a,b]$ $(1 < p < \infty)$. Then the Hardy-Littlewood majorant H_f of f belongs to $L_p[a,b]$ and

$$||H_f||_{L_p[a,b]} \leq 2^{1/p} \frac{p}{p-1} ||f||_{L_p[a,b]}.$$

Now we state a standard inequality which qualitatively bounds an intermediate derivative in terms of the highest derivative and the function, in L_p-norms.

Lemma 0.6.3. Let $f \, \varepsilon \, L_p[a,b]$ $(1 \leq p < \infty)$ be such that $f^{(k)} \, \varepsilon \, A.C. \, [a,b]$ and $f^{(k+1)} \, \varepsilon \, L_p[a,b]$. Then

$$||f^{(j)}||_{L_p[a,b]} \leq M_j \{||f^{(k+1)}||_{L_p[a,b]} + ||f||_{L_p[a,b]}\},$$

$j=1,2,\ldots,k$, where M_j's are certain constants depending only on $j,k,p,$ a and b.

We shall also have occasions to use Riesz-Thorin's interpolation theorem [11 , p. 2] which we state in

Lemma 0.6.4. Let $1 \leq p_0, q_0, p_1 (\neq p_0), q_1 (\neq q_0) \leq \infty$ and $T: L_{p_i}[a,b] \to L_{q_i}[c,d]$ be a bounded linear map with norm M_i $(i = 0,1)$. Then $T: L_p[a,b] \to L_q[c,d]$ with norm $M \leq M_0^{1-\theta} M_1^{\theta}$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \, , \, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \text{ and } 0 < \theta < 1.$$

Next we give some estimates about Steklov means in terms of integral modulii of smoothness of functions.

Lemma 0.6.5 [61] . Let $f \in L_p[a,b]$ $(1 \leq p < \infty)$ and $[a_1,b_1] \subset (a,b)$. Then, for all sufficiently small $\eta > 0$, $f_{\eta,m}$ has derivatives upto order m, the (m-1)th derivative is absolutely continuous over $[a_1,b_1]$ and the mth derivative exists a.e. on $[a_1,b_1]$ and belongs to $L_p[a_1,b_1]$. Moreover there hold:

$$||f_{\eta,m}^{(r)}||_{L_p[a_1,b_1]} \leq M_r \, \eta^{-r} \, \omega_r(f,\eta,p,[a,b]) \quad r = 1,2,\ldots,m,$$

$$||f-f_{\eta,m}||_{L_p[a_1,b_1]} \leq M_{m+1} \, \omega_m(f,\eta,p\,[a,b]),$$

$$||f_{\eta,m}||_{L_p[a_1,b_1]} \leq M_{m+2} \, ||f||_{L_p[a,b]},$$

$$||f_{\eta,m}^{(m)}||_{L_p[a_1,b_1]} \leq M_{m+3} \, \eta^{-m} ||f||_{L_p[a,b]},$$

where the constants M_i's are independent of f and η.

An inequality relating a lower order integral modulus of smoothness with a higher order integral modulus of smoothness is given in

Lemma 0.6.6 [65] . Let $f \in L_p[a,b]$ $(1 \leq p < \infty)$. Then

$$\omega_k(f,t,p,[a,b]) \leq M_k \, t^k \, \{||f||_{L_p[a,b]} + \int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} \, du \, \},$$

where the constant M_k is independent of f.

We close this section by stating the well-known Minkowski inequality.

Lemma 0.6.7. If the function $f(x,y)$ is measurable on the rectangle $a \leq x \leq b$, $c \leq y \leq d$ and for some $q \geq 1$, we have

$$\int_c^d \left\{ \int_a^b |f(x,y)|^q \, dx \right\}^{1/q} dy < \infty, \text{ then}$$

$$\left\{ \int_a^b \left| \int_c^d f(x,y) \, dy \right|^q dx \right\}^{1/q} \leq \int_c^d \left\{ \int_a^b |f(x,y)|^q \, dx \right\}^{1/q} dy.$$

0.7 A survey of the contents of the thesis

The thesis consists of five main chapters (excluding the present chapter 0). It is a study of local L_p-inverse theorems ($O(\varphi)$ and $o(\varphi)$) of a general order φ, for linear combinations of linear positive operators U_n, P_n, T_λ, S_n and $L_{n,p}$. Interpolatory modifications of the operators P_n and S_n are also considered. In each chapter we deal with one sequence or class of operators. A chapterwise summary of the thesis is as follows:

Chapter I: This is a study of the L_p-approximation ($1 \leq p < \infty$) by linear combinations $U_n(\cdot, k, t)$ of the operators U_n. Many well-known operators like Gauss-Weierstrass operators, De la Vallee Poussin Operators, Landau operators etc. are particular cases of these operators. Section 2 is a brief introduction about the general order φ of approximation under consideration. The results in this section are useful in proving all our φ-inverse theorems and contain a generalisation of a lemma of Berens and Lorentz [10] and another of Timan [65] . Section 3 contains some basic results

about the operators U_n and $U_n(.,k,t)$. Lemma 1.3.5, in
this section, is a localisation lemma, which implies that
the degree of approximation by U_n or their linear
combinations essentially depends on the behaviour of β in
a neighbourhood of origin. In section 4 we prove a direct
theorem for the linear combinations $U_n(.,k,t)$. Sections 5
and 6 are devoted to our main results viz., the $O(\varphi)$ and
$o(\varphi)$-inverse theorems. These are local in nature in the set
up of contracting subintervals; so are the $O(\varphi)$ and $o(\varphi)$
inverse theorems in subsequent chapters.

Chapter II: In this chapter we study φ-inverse theorems
for linear combinations $P_n(.,k,t)$ and interpolatory
modifications $P_{n,m}(.,t)$ of Bernstein-Kantorovitch polynomials
$P_n(.,t)$. Section 1 gives a brief introduction about the
operators and Section 2 consists of basic results about the
operators $P_n(.,t)$, $P_n(.,k,t)$ and $P_{n,m}(.,t)$. The $O(\varphi)$
and $o(\varphi)$-inverse theorems for linear combinations $P_n(.,k,t)$
are obtained in Sections 3 and 4, respectively. Lastly, in
Sections 5 and 6 we prove $O(\varphi)$ and $o(\varphi)$-inverse theorems for
the operators $P_{n,m}(.,t)$.

Chapter III. This chapter deals with linear combinations
$T_\lambda(.,k,t)$ of generalised Müller operators T_λ, which generalise
several well-known operators including the Gamma operators of
Müller [43] and Post-Widder [66] and modified Post-
Widder operators [39]. The operators T_λ are defined

in Section 1. Basic results about T_λ, which are useful in later sections, are dealt with in Section 2 and in Section 3, we prove a direct theorem. To prove this, we first obtain a lemma which estimates the error in approximation in terms of derivatives of a function (lemma 3.3.2). Section 4 contains the $O(\varphi)$-inverse theorem for linear combinations $T_\lambda(.,k,t)$ of the operators $T_\lambda(.,t)$. Lastly, in Section 5, we prove $o(\varphi)$-inverse theorem for the operators $T_\lambda(.,k,t)$.

Chapter IV. This chapter is a study of linear combinations $S_n(.,k,t)$ and interpolatory modifications $S_{n,m}(.,t)$ of regular exponential type operators. Section 1 is an introduction to the operators and Section 2 consists of basic results about the operators $S_n(.,t)$, $S_{n,m}(.,t)$ and $\underline{S}_n(.,k,t)$. Sections 3 and 4 are devoted, respectively, to the $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operator $S_n(.,k,t)$ and the last two sections, Sections 5 and 6, contain the $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operators $S_{n,m}(.,t)$.

Chapter V. This is the last chapter of the thesis dealing with the linear combinations $L_{n,p}(.,k,t)$ of the generalised Jackson operators $L_{n,p}(.,t)$ for functions which are 2π-periodic and belong to $L_q[-\pi,\pi]$ $(1 \le q < \infty)$. The operators and their linear combinations are described in Section 1. Section 2 contains some basic results on the moments, the convergence and the L_p-boundedness of the operators. Section 3 is a direct theorem for $L_{n,p}(.,k,t)$.

The last two sections(Sections 4 and 5) are devoted to $O(\varphi)$ and $o(\varphi)$-inverse theorems. It may be pointed out here that for $L_{n,p}(\cdot,k,t)$ we work with the order $\varphi(n^{-1})$ rather than the orders $\varphi(n^{-1/2})$ and $\varphi(\lambda^{-1/2})$ of earlier chapters. This is because the mth absolute moment of the operators $L_{n,p}(\cdot,t)$ is of order $O(n^{-m})$ whereas in the case of the operators of earlier chapters the mth absolute moments were of the order $O(n^{-m/2})$ and $O(\lambda^{-m/2})$. Lastly, we obtain global φ-inverse theorems for $L_{n,p}(\cdot,k,t)$ as a consequence of the local φ-inverse theorems.

CHAPTER I

φ-INVERSE THEOREMS FOR LINEAR COMBINATIONS OF OPERATORS U_n

1.1 Introduction.

In this chapter we study the linear combinations $U_n(\cdot, k, t)$ of a general class of convolution operators $U_n(\cdot, t)$ generated by powers of bell-shaped functions.

A function $\beta \in L_1(\mathbb{R})$ is said to be bell-shaped if

(i) $\beta(t) \geq 0$ for all $t \in \mathbb{R}$

(ii) $\beta(t)$ is continuous at $t = 0$ and $\beta(0) = 1$

(iii) $\sup\limits_{|t| \geq \delta} \beta(t) < 1$ for each $\delta > 0$.

The class of all bell-shaped functions will be denoted by B. By $B^{\infty,2}$ we mean the class of $\beta \in B$ which are infinitely differentiable in a neighbourhood of the origin and for which $\beta^{(2)}(0) \neq 0$. Also, for any $\delta > 0$, B_δ denotes the class of all $\beta \in B^{\infty,2}$ such that supp $\beta \subset (-\delta, \delta)$ and β is infinitely differentiable on $(-\delta, \delta)$.

With $\beta \in B$, the class $\{U_n : n \geq 1\}$ of linear positive convolution operators U_n is defined by

(1.1.1) $U_n(f; t) = \dfrac{1}{\alpha(n)} \displaystyle\int_{-\infty}^{\infty} \beta^n(t-u) \ f(u) \ du$

where

(1.1.2) $\alpha(n) = \displaystyle\int_{-\infty}^{\infty} \beta^n(u) \ du.$

Several approximation operators [58] can be obtained
as particular cases of the operators U_n. Here are some examples :

1. If

$$\beta(t) = e^{-\frac{t^2}{2}}, \ t \ \varepsilon \ \mathbb{R}$$

we get the Gauss-Weierstrass operators W_n defined by

$$W_n(f;t) = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{n}{2}(u-t)^2} f(u) \ du$$

2. If

$$\beta(t) = \{ \begin{array}{ll} \cos^2 t/2 \ , & t \ \varepsilon \ [-\pi,\pi] \\ 0 \ , & t \ \varepsilon \ \mathbb{R} - [-\pi,\pi], \end{array}$$

we get De la Vallée-Poussin operators V_n on the space $M_{2\pi}$
of all 2π-periodic, bounded, complex-valued measurable
functions on $\mathbb{R}$, defined by

$$V_n(f;t) = \frac{(2n)!!}{2\pi(2n-1)!!} \int_{-\pi}^{\pi} \cos^{2n} (\frac{u-t}{2}) \ f(u) \ du \ .$$

3. For $a,b,c \ \varepsilon \ \mathbb{R}$ such that $c \geq b-a > 0$,

$$\beta(t) = \{ \begin{array}{ll} 1 - (\frac{t}{c})^2 & , \ |t| \leq c \\ 0 & , \ |t| > c \ , \end{array}$$

gives the Landau operators L_n on the space of all bounded, complex
valued, measurable functions vanishing outside the interval
(a,b) defined by

$$L_n(f;t) = \frac{1}{I(n)} \int_{a}^{b} [(\frac{c^2 - (u - t)^2}{c^2})]^n \ f(u) \ du \ ,$$

where

$$I(n) = \int_{-c}^{c} (\frac{c^2-u^2}{c^2})^n \ du .$$

Operators of the type $U_n(.,t)$ were considered for the first time by Korovkin [31, chapter 1]. Later, Sikkema and Rathore [58] obtained results on the convergence in the basic approximation of continuous functions and under the assumption that $\beta''(0) \neq 0$, formulae of Voronvskaja type, giving a precise rate of this convergence for IInd differentiable functions, Lipschitz-Nikolskii constants and convergence in the simultaneous approximation of derivatives of functions with the corresponding derivatives of the operators. Next, Sikkema [59] constructed a subclass B' of B which yields a much higher order of approximation than any power of n. Also, he [69] obtained asymptotic formulae of Voronovskaya type for the speed with which $U_n(f;t) \to 0$ as $n \to \infty$.

Very recently, Winslin [67] has obtained global direct, inverse and saturation theorems for the linear combinations in L_p-norm $(1 \leq p < \infty)$. His inverse theorems correspond to the orders $n^{-\alpha/2}$ of approximation $(0 < \alpha < 2k+2)$.

Our aim here is to study the local approximation by the linear combinations $U_n(.,k,t)$ of the convolution operators $U_n(.,t)$ for $f \varepsilon L_p(\mathbb{R})$. It is clear that $U_n f$ is meaningful for $f \varepsilon L_p(\mathbb{R})$ and moreover $||U_n f||_{L_p(-\infty,\infty)} \leq ||f||_{L_p(-\infty,\infty)}$. Further, for $f \varepsilon L_p(-\infty,\infty)$, $||U_n f-f||_{L_p(\mathbb{R})} \to 0$ as $n \to \infty$ $(1 \leq p < \infty)$.

Let $d_0,...,d_k$ be k+1 distinct positive numbers. Then for $n \geq \max_{0 \leq i \leq k} d_j^{-1}$, the linear combinations $U_n(f,k,t)$ are defined by

$$U_n(f,k,t) = \sum_{j=0}^{k} C(j,k) \; U_{d_j n}(f;t),$$

where

$$C(j,k) = \begin{cases} \prod_{\substack{i=0 \\ i \neq j}}^{k} \dfrac{d_j}{d_j - d_i} & \text{if } k \neq 0 \\[2mm] 1 & \text{if } k = 0 \, . \end{cases}$$

In this chapter, for the operators $U_n(.,k,t)$ with $\beta \; \varepsilon \; B^{\infty,2}$, we obtain local direct and inverse theorems of a more general order $\varphi(n^{-1/2})$ in L_p-norm $(1 \leq p < \infty)$ over contracting subintervals. All through this chapter, unless stated otherwise, the operators $U_n(.,t)$ correspond to $\beta \; \varepsilon \; B^{\infty,2}$. It may be remarked here that our results corresponding to $k = C$ constitute local approximation by the operators U_n themselves.

In the sequel, Section 1.2 introduces the general order φ of approximation and Section 1.3 contains basic results about the operators $U_n(.,t)$ and $U_n(.,k,t)$. The local direct theorem about $U_n(.,k,t)$ is proved in Section 1.4. Lastly, in Sections 1.5 and 1.6, respectively, we obtain our main results, viz., $O(\varphi)$ and $o(\varphi)$-inverse theorems.

1.2   The order φ of approximation.

This section consists exclusively of the results about the order φ of approximation. The results here include the φ-generalisation of a lemma of Berens and Lorentz [12] .

Definition 1.2.1.

For any $r \; \varepsilon \; \mathbb{N}$, a function φ on $(0,c)$ is said to belong to the class Φ_r if it satisfies the following :

1. For any $0 < h < 1$, there exists a constant $K_\varphi(h)$ depending only on φ and h such that, for all $t \in (0,c]$,

$$\frac{\varphi(t)}{\varphi(th)} \leq K_\varphi(h)$$

2. For any $\delta > 0$,

$$K_\delta = \sup_{h \geq \delta} K_\varphi(h) < \infty$$

and lastly,

3. $h^r K_\varphi(h) \to 0$ as $h \to 0$.

We give some illustrative examples of $\varphi \in \Phi_r$ at the end of this section.

Lemma 1.2.2. Let $c > 0$, Ω be an increasing function on $(0,c]$ and for some $r \in \mathbb{N}$, $\varphi \in \Phi_r$ such that, for all $t,h \in (0,c]$,

(1.2.1) $\qquad \Omega(h) \leq M\{ \varphi(t) + (\frac{h}{t})^r \Omega(t) \}$.

Then

(1.2.2) $\qquad \Omega(t) = O(\varphi(t))$ $(t \to 0)$.

Remark 1. The aforementioned lemma of Berens and Lorentz [10] is a special case of this lemma obtained by taking $\varphi(t) = t^\alpha (0 < \alpha < r)$.

Remark 2. This lemma would be extensively used here and in subsequent chapters, in the proofs of $O(\varphi)$-inverse theorems.

Proof of Lemma 1.2.2. Since $h^r K_\varphi(h) \to 0$ as $h \to 0$, choose $A > 1$ such that $A^{-r} K_\varphi(A^{-1}) < \frac{1}{2M}$. Let $h_m = cA^{1-m}$ and

$$M_1 = \max \left\{ \frac{\bar{\Omega}(c)}{\varphi(c)} , \ 2M \, K_\varphi(A^{-1}) \right\}.$$

First, we prove that

(1.2.3) $\qquad \Omega(h_m) \leq M_1 \, \varphi(h_m) \quad$ for $m = 1,2,\ldots$

by induction on m.

Since $h_1 = c$, (1.2.3) is true for $m = 1$.

Assuming (1.2.3) for $m-1$ and taking $h = h_m$ and $t = h_{m-1}$ in (1.2.1),

$$\Omega(h_m) \leq M \left\{ \varphi(h_{m-1}) + A^{-r} \, \Omega(h_{m-1}) \right\}$$

$$\leq M \left\{ K_\varphi(A^{-1}) + A^{-r} M_1 \, K_\varphi(A^{-1}) \right\} \varphi(h_m)$$

$$\leq M_1 \, \varphi(h_m).$$

Thus (1.2.3) is proved for m and hence is true for $m = 1,2,\ldots$

Now, $t \, \varepsilon \, (0,c]$ implies that $h_m < t \leq h_{m-1}$ for some $m \, \varepsilon \, \mathbb{N}$. Hence,

$$\Omega(t) \leq \Omega(h_{m-1}), \text{ which from (1.2.3) becomes}$$

$$\leq M_1 \, \varphi(h_{m-1})$$

$$\leq M_1 \, K_\varphi\left(\frac{t}{h_{m-1}}\right) \varphi(t)$$

$$\leq M K_{A^{-1}} \, \varphi(t),$$

since $\dfrac{t}{h_{m-1}} \geq \dfrac{h_m}{h_{m-1}} = A^{-1}$ and hence the lemma is proved.

The next lemma would be of crucial importance in the proofs of $o(\varphi)$ inverse theorems.

Lemma 1.2.3.   Let $c > 0$, $\varphi \in \Phi_r$ for some $r \in \mathbb{N}$ and $\Omega(t)$ be a positive increasing function on $(0,c]$ such that, for all $t, h \in (0,c]$,

$$(1.2.4) \qquad \Omega(t) \leq M\{\psi(t) + (\tfrac{h}{t})^r \Omega(t)\},$$

where

$$(1.2.5) \qquad \psi(t) = o(\varphi(t)) \quad (t \to 0).$$

Then

$$\Omega(t) = o(\varphi(t)) \quad (t \to 0).$$

Proof.   From (1.2.5), $\psi(t) = e(t) \varphi(t)$, where

$$e(t) \to 0 \text{ as } t \to 0.$$

Since $\varphi \in \Phi_r$, $t^r K_\varphi(t) \to 0$ as $t \to 0$.   Hence, we can choose $A > 1$ such that

$$A^{-r} K_\varphi(A^{-1}) < \frac{1}{4M} .$$

Let $h_m = cA^{1-m}$, $\delta_m = \dfrac{\Omega(h_m)}{\varphi(h_m)}$ and $\varepsilon > 0$.

Then, there exists $n_0 \in \mathbb{N}$ such that, for all $m \geq n_0$,

$$e(h_{m-1}) K_\varphi(A^{-1}) M < \varepsilon/4.$$

Taking $h = h_m$ and $t = h_{m-1}$ in (1.2.4) we get

$$\Omega(h_m) \leq M\{\psi(h_{m-1}) + A^{-r} \Omega(h_{m-1})\}$$

$$= M\{e(h_{m-1}) \varphi(h_{m-1}) + A^{-r} \delta_{m-1} \varphi(h_{m-1})\}$$

$$\leq M\{e(h_{m-1}) K_\varphi(A^{-1}) \varphi(h_m) + A^{-r} K_\varphi(A^{-1}) \delta_{m-1} \varphi(h_m)\}$$

$$< (\varepsilon/4 + \frac{\delta_{m-1}}{4}) \varphi(h_m)$$

and hence

$$\delta_m < \varepsilon/4 + \frac{\delta_{m-1}}{4}.$$

Thus, proceeding recursively,

$$\delta_{m+n_o} < \varepsilon/4 + \varepsilon/4^2 + \ldots + \varepsilon/4^m + \frac{\delta_{n_o}}{4^m}$$

$$= (\varepsilon/3)(1 - \frac{1}{4^m}) + \frac{\delta_{n_o}}{4^m} .$$

Since ε is arbitrary, we get that $\delta_m \to 0$ as $m \to \infty$, which implies that

$$\Omega(h_m) = o(\varphi(h_m)) \quad (m \to \infty)$$

Now, $t \varepsilon (0,c]$ implies that there exists $m \varepsilon \mathbb{N}$ such that $h_m < t \leq h_{m-1}$. Thus

$$\Omega(t) \leq \Omega(h_{m-1}) = \delta_{m-1} \varphi(h_{m-1})$$

$$\leq \delta_{m-1} K_\varphi(t/h_{m-1}) \varphi(t)$$

$$\leq \delta_{m-1} K_{A^{-1}} \varphi(t) \qquad (\because t/h_{m-1} \geq A^{-1}).$$

Therefore,

$$\Omega(t) = o(\varphi(t)) \qquad (t \to 0).$$

Lemma 1.2.4. Let $k \varepsilon \mathbb{N}$, $f \varepsilon L_p[a,b]$ $(1 \leq p < \infty)$ and $\varphi \varepsilon \Phi_k$ such that

$$(1.2.6) \qquad \omega_{k+1}(f,t,p,[a,b]) = O(\varphi(t)) \qquad (t \to 0).$$

Then

$$\omega_k(f,t,p,[a,b]) = O(\varphi(t)) \qquad (t \to 0).$$

<u>Remark</u> : This lemma is a φ-generalisation of the following well-known result (see $[65]$, $\S 3.34, \S 3.3.12\text{-}13,\text{pp. }110\text{-}12]$ and $[61]$). In the sequel we put $\Phi(t) = \Phi(c)$, $t \, \varepsilon \, (c,\infty)$.

<u>Proof of the lemma 1.2.4</u> : By hypothesis (1.2.6), we have

$$\int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} \, du \leq M_1 \int_t^{\frac{b-a}{2k}} \varphi(u) \, u^{-k-1} \, du$$

and hence choosing $h(0 < h < 1)$ such that $h^r K_\varphi(h) < 1$, we get

$$\int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} \, du \leq M_1 \sum_{P=0}^{\infty} \int_{t/h^P}^{t/h^{P+1}} \varphi(u) \, u^{-k-1} \, du$$

$$\leq M_1 \sum_{P=0}^{\infty} \varphi(t/h^P) \int_{t/h^P}^{t/h^{P+1}} \frac{\varphi(u)}{\varphi(t/h^P)} \, u^{-k-1} \, du$$

$$\leq M_1' \, K_h \sum_{P=0}^{\infty} \varphi(t/h^P) \int_{t/h^P}^{t/h^{P+1}} u^{-k-1} \, du,$$

since $t/uh^P \geq h$. Thus,

$$\int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} \, du \leq \frac{M_2}{t^k} \sum_{P=0}^{\infty} \varphi\left(\frac{t}{h^P}\right) h^{Pk}.$$

But,

$$\frac{\varphi(t/h^P)}{\varphi(t)} = \frac{\varphi(t/h^P)}{\varphi(t/h^{P-1})} \, \frac{\varphi(t/h^{P-1})}{\varphi(t/h^{P-2})} \cdots \frac{\varphi(t/h)}{\varphi(t)} \leq (K_\varphi(h))^P .$$

Hence, using the fact that $K_\varphi(h) \, h^k < 1$,

$$\int_t^{\frac{b-a}{2k}} \omega_{k+1}(f,u,p,[a,b]) \, du \leq \frac{M_3}{t^k} \varphi(t) .$$

Using a result of [65] , viz., for any $f \in L_p[a,b]$ $(1 \le p < \infty)$,

(1.2.7) $\omega_k(f,t,p,[a,b])$

$$\le M_k \, t^k \{ ||f||_{L_p[a,b]} + \int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} \, du \},$$

we have

(1.2.8) $\omega_k(f,t,p,[a,b]) \le M_k \, t^k \{ ||f||_{L_p[a,b]} + \frac{M_3}{t^k} \varphi(t) \}$.

Now, since $\varphi \in \Phi_k$, $h^k K_\varphi(h) \to 0$ as $h \to 0$. Thus, we get that

$$t^k / \varphi(t) \to 0 \text{ as } t \to 0.$$

Hence, (1.2.8) implies that

$$\omega_k(f,t,p,[a,b]) = O(\varphi(t)) \ (t \to 0).$$

The little -o version of lemma 1.2.4 is

Lemma 1.2.5 :   Let $k \in \mathbb{N}$, $f \in L_p[a,b]$ $(1 \le p < \infty)$ and $\varphi \in \Phi_k$ such that

$$\omega_{k+1}(f,t,p,[a,b]) = o(\varphi(t)) \ (t \to 0) .$$

Then

$$\omega_k(f,t,p,[a,b]) = o(\varphi(t)) \ (t \to 0).$$

Proof :   Let $\varepsilon > 0$. Since $\omega_{k+1}(f,t,p,[a,b]) = o(\varphi(t))$, we can choose a sufficiently small $\delta > 0$ such that, for all $t \in (0,\delta]$,

$$\omega_{k+1}(f,t,p,[a,b]) \le \varepsilon \varphi(t).$$

Hence,

$$\int_t^\delta \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} \, du \le \varepsilon \int_t^\infty \frac{\varphi(u)}{u^{k+1}} \, du .$$

Thus, proceeding as in the proof of lemma1.2.4, we get,

$$(1.2.9) \qquad \int_t^\delta \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} \, du \leq \frac{\varepsilon M_3}{t^k} \, \varphi(t).$$

Next,

$$(1.2.10) \qquad \int_\delta^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} \, du \leq \omega_{k+1}\left(f, \frac{b-a}{2k}, p, [a,b]\right) \frac{\delta^{-1}}{k}$$

$$= M_4, \text{ say.}$$

Now, from (1.2.9) and (1.2.10),

$$\int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} du \leq \frac{\varepsilon M_3}{t^k} \, \varphi(t) + M_4.$$

Thus, from (1.2.7),

$$\omega_k(f,t,p,[a,b]) \leq M_k t^k ||f||_{L_p[a,b]}$$

$$+ M_k t^k \int_t^{\frac{b-a}{2k}} \frac{\omega_{k+1}(f,u,p,[a,b])}{u^{k+1}} \, du$$

$$\leq M_k t^k ||f||_{L_p[a,b]} + M_3' \varepsilon \, \varphi(t) + M_4 M_k t^k$$

$$= M_k' t^k ||f||_{L_p[a,b]} + M_k'' \varepsilon \, \varphi(t), \text{ say.}$$

Hence,

$$\frac{\omega_k(f,t,p,[a,b])}{\varphi(t)} \leq M_k' \frac{t^k}{\varphi(t)} ||f||_{L_p[a,b]} + M_k'' \varepsilon.$$

Thus, using the fact that

$$t^k/\varphi(t) \to 0 \text{ as } t \to 0, \text{ we get}$$

$$\lim_{t \to 0} \sup \frac{\omega_k(f,t,p,[a,b])}{\varphi(t)} \leq \varepsilon M_k''.$$

Since ε is arbitrary, we get the result.

Examples of approximation orders $\varphi \ \varepsilon \ \Phi_r$:

Throughout our discussion in the rest of this section, we take $r \ \varepsilon \ \mathbb{N}$, $\alpha \ \varepsilon \ \mathbb{R}$ such that $0 < \alpha < r$.

1. For $t \ \varepsilon \ (0,c]$, $c > 0$,

    $$\varphi(t) \ = \ t^\alpha \ \varepsilon \ \Phi_r.$$

    For, here we can take $\overline{K}_\varphi(h) = \dfrac{1}{h^\alpha}$. Then clearly the other conditions are also satisfied.

2. Let $p \ \varepsilon \ \mathbb{N}$, $q > 0$ and $c > 0$ be such that $\log^p 1/c > 0$. Then, for $t \ \varepsilon \ (0,c]$,

    $$\varphi(t) \ = \ t^\alpha (\log^p 1/t)^q \ \ \varepsilon \ \Phi_r.$$

    For, here

    $$\sup_t \ \frac{\varphi(t)}{\varphi(th)} \leq 1/h^\alpha$$

    and hence, taking $K_\varphi(h) = 1/h^\alpha$, we observe that the other two conditions of Φ_r are also satisfied.

3. Let $p \ \varepsilon \ \mathbb{N}$, $q > 0$ and $c > 0$ be such that $\log^p 1/c > 0$. Then, for

    $$\varphi(t) \ = \ t^\alpha/(\log^p 1/t)^q \ \ \ , \ t \ \varepsilon \ (0,c] \ ,$$

    $$\overline{K}_\varphi(h) \ = \ \sup_t \frac{\varphi(t)}{\varphi(th)} = \frac{1}{h^\alpha} \ \left(\frac{\log^p 1/ch}{\log^p 1/c}\right)^q$$

    and hence $\varphi \ \varepsilon \ \Phi_r.$

4. Let $-\infty \leq a < b \leq \infty$, $k \in \mathbb{N}$. Let $f \in L_p [a,b]$ $(1 \leq p \leq \infty)$ be a function which is not a polynomial of degree $\leq k-1$. Then, for r $(> k)$ $\in \mathbb{N}$,

$$\varphi(t) = \omega_k(f,t,p,[a,b]) \in \Phi_r$$

where $t \in (0,c]$, $c < \frac{b-a}{2k}$. For, by (Timan [65]),

$$\frac{\varphi(t)}{\varphi(th)} \leq (1 + \frac{1}{h})^k = K_\varphi(h)$$

and $K_\varphi(h)$ satisfies the required conditions.

1.3. Basic Results

In this section we give some basic estimates and results about the operators $U_n(.,t)$ and $U_n(.,k,t)$, which will be used in the later sections.

Lemma 1.3.1 [58] : Let $k \in \mathbb{N}^o$ and $\beta \in B$. Then

$$\lim_{n \to \infty} \frac{\sum\limits_{i=0}^{k} (-1)^i \binom{k}{i} \alpha(n+i+1)}{\sum\limits_{i=0}^{k} (-1)^i \binom{k}{i} \alpha(n+i)} = 1 \, ,$$

where $\alpha(n)$ is given by (1.1.2).

Taking $k = 0$ in lemma 1.3.1 and applying it repeatedly we obtain

Corollary 1.3.2 [58] : For $m \in \mathbb{N}$,

$$\lim_{n \to \infty} \frac{\alpha(n-m)}{\alpha(n)} = 1 \, ,$$

where $\alpha(n)$ is given by (1.1.2).

The following lemma gives an asymptotic expansion of the moments of the operators $U_n(\cdot, t)$.

Lemma 1.3.3   Let $\delta > 0$, $\beta \varepsilon B_\delta$ and $k \varepsilon \mathbb{N}$.   Then the following asymptotic expansion is valid:

$$\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} t^k \beta^n(t) dt \sim \sum_{r=[\frac{k+1}{2}]}^{\infty} \frac{C_{k,r}}{n^r} , \quad (n \to \infty)$$

where $C_{k,r}$ are constants independent of n.

Proof: Assuming that $\beta \varepsilon B$ is m-times continuously differentiable in a neighbourhood of the origin with $\beta''(0) \neq 0$ and $t^k \beta^\ell(t) \varepsilon L_1(-\infty, \infty)$ for some $\ell \geq 1$, Winslin [67] has shown that

$$\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} t^k \beta^n(t) dt = \sum_{r \geq [\frac{k+1}{2}]} \frac{C_{k,r}}{n^r} + o(\frac{1}{n^{\frac{k+m-2}{2}}}),$$

where $C_{k,r}$ do not depend on n.

Since $\beta \varepsilon B_\delta$, these assumptions are satisfied for an arbitrary m and hence the lemma.

Corollary 1.3.4   Let $k \varepsilon \mathbb{N}$, $\delta > 0$ and $\beta \varepsilon B_\delta$. Then

$$U_n(|u-t|^k ; t) \leq \frac{M}{n^{k/2}} , \quad \text{M being a constant.}$$

Proof   Let $s > k$ be an even integer.   Then by Hölders inequality

$$\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u) |u-t|^k du$$

$$\leq \{\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u) |u-t|^s du\}^{k/s} \times$$

$$\times \{\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u) du\}^{1-k/s}$$

Now the lemma follows from lemma 1.3.3.

The following lemma has a crucial importance all over this chapter. It is a tool with which we can extend our results for any $\beta \in B^{\infty,2}$ by just proving them for $\beta \in B_{\delta}$ for an appropriate $\delta > 0$.

<u>Lemma</u> 1.3.5   Let $\beta, \beta^* \in B$ such that $\beta = \beta^*$ on $(-\delta, \delta)$ for some $\delta > 0$. Then for any $\ell > 0$, and $f \in L_p(-\infty, \infty)$ $(1 \leq p < \infty)$,

$$||U_n(f, t) - U^*_n(f, t)||_{L_p(-\infty, \infty)} = O(n^{-\ell}) \, ||f||_{L_p(-\infty, \infty)} \quad (n \to \infty)$$

where U_n, U^*_n are the operators corresponding to β and β^* respectively.

<u>Proof</u> $\int_{-\infty}^{\infty} \beta^n(u) \, f(t-u) \, du = \int_{-\delta}^{\delta} \beta^n(u) f(t-u) \, du + \int_{|u| \geq \delta} \beta^n(u) f(t-u) \, du$

$$= f_n(t) + g_n^{\beta}(t), \text{ say.}$$

Then, assuming that $X_{\delta}(u)$ is the characteristic function of $\mathbb{R} - (-\delta, \delta)$, we have

$$(1.3.1) \quad ||g_n^{\beta}(t)||_{L_p(-\infty, \infty)} \leq \int_{-\infty}^{\infty} X_{\delta}(u) \beta^n(u) \, ||f||_{L_p(-\infty, \infty)} \, du$$

$$\leq \alpha_{\beta}(1) \, \beta_{\delta}^{n-1} \, ||f||_{L_p(-\infty, \infty)},$$

where

$$\beta_{\delta} = \sup_{\mathbb{R} - (-\delta, \delta)} \beta(t) < 1 \qquad \text{and}$$

$$\alpha_{\beta}(1) = \int_{-\infty}^{\infty} \beta(u) \, du.$$

Write

$$\alpha(n) = \alpha_{\beta}(n) = \int_{-\delta}^{\delta} \beta^n(u) \, du + \int_{|u| \geq \delta} \beta^n(u) \, du$$

$$= \gamma_n + \delta_n^{\beta}, \qquad \text{say.}$$

Clearly

$$(1.3.2) \qquad \delta_n^{\beta} \leq \alpha_{\beta}(1) \ \beta_{\delta}^{n-1} \ .$$

Let $\delta^* \leq \delta$ be such that, for $|u| < \delta^*$,

$$(1.3.3) \qquad \beta(u) > \max \{ \frac{1+\beta_{\delta}}{2} , \frac{1+\beta_{\delta}^*}{2} \} = \lambda , \qquad \text{say.}$$

Then

$$(1.3.4) \qquad \lambda > \max \{ \beta_{\delta}^*, \ \beta_{\delta} \} \qquad \text{and}$$

$$(1.3.5) \qquad \gamma_n \geq 2\delta \ \lambda^n > 2\delta^* \lambda^n \ .$$

Hence

$$||U_n f - U_n^* f||_{L_p(-\infty,\infty)} = || \frac{f_n + g_n^{\beta}}{\gamma_n + \delta_n^{\beta}} - \frac{f_n + g_n^{\beta^*}}{\gamma_n + \delta_n^{\beta^*}} ||_{L_p(-\infty,\infty)}$$

$$\leq \frac{1}{\alpha_{\beta}(n) \cdot \alpha_{\beta^*}(n)} \{ ||f_n||_{L_p(-\infty,\infty)} (\delta_n^{\beta} + \delta_n^{\beta^*})$$

$$+ \alpha_{\beta^*}(n) ||g_n^{\beta}||_{L_p(-\infty,\infty)} + \alpha_{\beta}(n) ||g_n^{\beta^*}||_{L_p(-\infty,\infty)}$$

Thus from $(1.3.1)$, $(1.3.2)$, $(1.3.5)$ and using the fact that $||f_n||_{L_p(-\infty,\infty)} \leq ||f||_{L_p(-\infty,\infty)}$, we get that

$$||U_n f - U_n^* f||_{L_p(-\infty,\infty)} \leq ||f||_{L_p(-\infty,\infty)} \{ \frac{\alpha_{\beta}(1) \beta_{\delta}^{n-1} + \alpha_{\beta^*}(1) \beta_{\delta}^{*n-1}}{2\delta^* \lambda^n} \}$$

$$+ \frac{||f||_{L_p(-\infty,\infty)}}{2\delta^* \lambda^n} \alpha_{\beta}(1) \beta_{\delta}^{n-1} + \frac{||f||_{L_p(-\infty,\infty)} \alpha_{\beta^*}(1) \beta_{\delta}^{*n-1}}{2\delta^* \lambda^n}$$

Now, since $\frac{\beta_{\delta}}{\lambda} < 1$, for any $\ell > 0$,

$$\frac{\alpha_{\beta}(1) \beta_{\delta}^{n-1}}{2\delta^* \lambda^n} = \frac{\alpha_{\beta}(1)}{2\delta^* \lambda} \cdot (\frac{\beta_{\delta}}{\lambda})^{n-1} = O(n^{-\ell})$$

Similarly, for any $\ell > 0$,

$$\frac{\alpha_\beta(1)\,\beta_\delta^{*n-1}}{2\,\delta^*\lambda^n} = O(n^{-\ell}) \quad \text{and hence}$$

$$||U_n t - U_n^* f||_{L_p(-\infty,\infty)} = ||f||_{L_p(-\infty,\infty)} \cdot O(n^{-\ell})$$

Hence the result.

Now we prove two localisation lemmas which would be found useful in the later sections.

Throughout the rest of this chapter, we use notation $I_j = [a_j, b_j]$, $j = 1,2$ where $-\infty < a_1 < a_2 < b_2 < b_1 < \infty$.

<u>Lemma</u> 1.3.6   Let $f \in L_p(-\infty,\infty)$ $(1 \le p < \infty)$ and $\beta \in B_\delta$ with $\delta < \min(a_2 - a_1, b_1 - b_2)$. Then, for any $\ell > 0$,

$$||U_n(f;t)||_{L_p(I_2)} \le ||f||_{L_p(I_1)} \cdot$$

<u>Proof</u>   By Jensen's inequality

$$||U_n(f;t)||_{L_p(I_2)}^p \le \frac{1}{\alpha(n)} \int_{a_2}^{b_2} \int_{-\infty}^{\infty} \beta^n(u)\,|f(t-u)|^p \, du\,dt$$

$$= \frac{1}{\alpha(n)} \int_{a_2}^{b_2} \int_{-\delta}^{\delta} \beta^n(u)\,|f(t-u)|^p \, du\,dt$$

$$\le \frac{1}{\alpha(n)} \int_{a_2-\delta-\delta}^{b_2+\delta} \int^{\delta} \beta^n(u)\,|f(t)|^p \, du\,dt$$

$$\le ||f||_{L_p(I_1)}^p$$

and hence the lemma.

The following lemma is clear from the definition of the operator $U_n(\cdot,t)$.

Lemma 1.3.7   Let a < c < d < b and X(u) be the character-
istic function of the interval [a,b],further, let

f ε $L_p(-\infty,\infty)$ (1 ≤ p < ∞) and β ε B_δ with δ < min(c-a,b-d).
Then, for all t ε [c,d]

$$U_n((1-X(u))f(u);t) = 0$$

Next we prove a lemma which gives an asymptotic expansion
of the error for functions in $C_0^{2k+2}(-\infty,\infty)$.

Lemma 1.3.8   Let f ε $C_0^{2k+2}(-\infty,\infty)$ and β ε B_δ for any δ > 0.
Then there holds

$$U_n(f,k,t)-f(t) = n^{-(k+1)}\{\sum_{i=1}^{2k+2} c_i f^{(i)}(t)+o(n^{-(k+1)}) (n \to \infty),$$

uniformly for t ε IR, where c_i's are certain constants.

Proof   For some ξ   lying between u and t we have

$$(1.3.6) \qquad f(u) = \sum_{i=0}^{2k+2} \frac{f^{(i)}(t)}{i!}(u-t)^i +$$

$$+ \frac{(u-t)^{2k+2}}{(2k+2)!} (f^{(2k+2)}(\xi)-f^{(2k+2)}(t)).$$

From lemma 1.3.3 and the fact that $\sum_{j=0}^{k} C(j,k) d_j^{-m}=0$, m=1,2,...,k,
it follows that for some constants c_i,

$$(1.3.7) \qquad U_n((u-t)^i, k,t) = n^{-(k+1)}c_i + o(n^{-(k+1)}) (n \to \infty),$$

where i=1,2,...,2k+2 and o-term is uniformly in t ε (-∞,∞).

Since $f^{(2k+2)}$ ε $C_0(-\infty,\infty)$, given an arbitrary ε > 0,]
δ > 0 such that whenever |x-y| < δ ,

$$|f^{(2k+2)}(x) - f^{(2k+2)}(y)| < \varepsilon.$$

Hence,

$$|U_n((u-t)^{2k+2}(f^{(2k+2)}(\xi)-f^{(2k+2)}(t));t)|$$

$$\leq \varepsilon|U_n(|u-t|^{2k+2},t)| + \frac{2}{\delta^2}||f^{(2k+2)}||_{C_0(-\infty,\infty)} \times$$

$$\times |U_n(|u-t|^{2k+4};t)|$$

$$\leq \varepsilon \cdot \frac{M}{n^{k+1}} + \frac{M_1}{n^{k+2}}$$

Thus

$$(1.3.8) \quad |U_n((u-t)^{2k+2}(f^{(2k+2)}(\xi) - f^{(2k+2)}(t));t)| \leq \frac{M_2}{n^{k+1}}(\varepsilon + \frac{1}{n}).$$

Now, combining (1.3.7) and (1.3.8) from (1.3.6) we get the result.

We close this section by stating a special case of a lemma of Sikkema and Rathore [58] .

Lemma 1.3.9   Let $\beta \in B_\delta$. Then for any $m < n$, there holds the identity:

$$\frac{\partial^m}{\partial u^m} \beta^n(u-t) = \beta^{n-m}(u-t) \sum_{k=0}^{m} \sum_{\nu=0}^{[\frac{m-k}{2}]} n^\nu (n \frac{\partial}{\partial u} \beta(u-t))^k g_{\nu,k,m}(u-t)$$

for all $t,u \in \mathbb{R}$, where $g_{\nu,k,m}$ are certain linear combinations of products of $\beta_1,\dots,\beta^{(m)}$ (which are independent of n) and hence bounded on $\mathbb{R}$.

1.4   Direct theorem

In this section we prove a direct theorem whose corresponding inverse theorem is dealt with in the next section.

<u>Theorem</u> 1.4.1   Let $f \varepsilon L_p(-\infty,\infty)$, $\beta \varepsilon B^{\infty,2}$ and $\varphi \varepsilon \Phi_{2k+2}$.
Then, for sufficiently large values of n

$$\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t)) \quad (t \to o)$$

implies that

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} = O(\varphi(\tfrac{1}{n})) \quad (n \to \infty).$$

Before proving the theorem, we prove some auxiliary
results which shall be used in the proof of the theorem.

<u>Lemma</u> 1.4.2   Let $\beta \varepsilon B_\delta$ with $\delta < \min(b_1-b_2, a_2-a_1)$ and
$f \varepsilon L_p(-\infty,\infty)$ $(1 \leq p < \infty)$ have $2k+2$ derivatives on I_1 with
$f^{(2k+1)} \varepsilon A.C.(I_1)$ and $f^{(2k+2)} \varepsilon L_p(I_1)$. Then

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} \leq \frac{M}{n^{k+1}} \{||f^{(2k+2)}||_{L_p(I_1)} + ||f||_{L_p(I_1)}\}$$

<u>Proof</u>   Assuming that $X(u)$ is the characteristic function of
I_1, we have from lemma 1.3.7, for all $t \varepsilon I_2$,

$$U_n(f;t) = \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u)X(u)f(u)du.$$

For $t \varepsilon I_2$ and $u \varepsilon I_1$, using

$$f(u) = \sum_{i=0}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{(2k+1)!} \int_{t}^{u}(u-w)^{2k+1}f^{(2k+2)}(w)dw,$$

we get,

$$U_n(f;t) = \frac{1}{\alpha(n)} \sum_{i=0}^{2k+1} \frac{f^{(i)}(t)}{i!} \int_{-\infty}^{\infty} \beta^n(t-u)X(u)(u-t)^i du$$

$$+ \frac{1}{\alpha(n)} \frac{1}{(2k+1)!} \int_{-\infty}^{\infty} \beta^n(t-u)X(u)\int_{t}^{u}(u-w)^{2k+1}f^{(2k+2)}(w)dw\,du.$$

Thus

$$U_n(f,k,t) - f(t) = \sum_{i=1}^{2k+1} \frac{f^{(i)}(t)}{i!} \{ \sum_{j=0}^{k} C(j,k) \frac{1}{\alpha(d_j n)} \int_{-\infty}^{\infty} \beta^{d_j n}(t-u) X(u) \times$$
$$\times (u-t)^i du$$

$$+ \frac{1}{(2k+1)!} \{ \sum_{j=0}^{k} C(j,k) \{ \frac{1}{\alpha(d_j n)} \int_{-\infty}^{\infty} \beta^{d_j n}(t-u) X(u) \times$$
$$\times \int_{t}^{u} (u-w)^{2k+1} f^{(2k+2)}(w) dw du \} \}$$

$$(1.4.1) \qquad = Z_1 + Z_2, \text{ say.}$$

Since $\sum_{j=0}^{k} C(j,k) d_j^{-m} = 0$, $m=1,2,\ldots,k$, using lemmas 1.3.3

we get

$$(1.4.2) \qquad ||Z_1||_{L_p(I_2)} \leq \frac{M}{n^{k+1}} \{ ||f^{(2k+2)}||_{L_p(I_2)} + ||f||_{L_p(I_2)} \}.$$

Consider

$$R = \int_{a_2}^{b_2} | \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} X(u) \beta^n(t-u) \int_{t}^{u} (u-w)^{2k+1} f^{(2k+2)}(w) dw|^p dt$$

which by Jensen's inequality is

$$\leq \frac{1}{\alpha(n)} \int_{a_2}^{b_2} \int_{a_1}^{b_1} \beta^n(t-u) |u-t|^{(2k+1)p} | \int_{t}^{u} X(u) f^{(2k+2)}(w) dw|^p du dt$$

$$\leq \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta^n(t) |t|^{(2k+1)p} | \int_{t+u}^{u} X(u) f^{(2k+2)}(w) dw|^p du dt.$$

Again using Jensen's inequality

$$R \leq \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t) |t|^{(2k+1)p} \int_{-\infty}^{\infty} \int_{t+u}^{u} |t|^{p-1} |X(w) f^{(2k+2)}|^p dw du dt$$

$$= \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t) |t|^{(2k+1)p} \int_{-\infty}^{\infty} \int_{w-t}^{w} |t|^{p-1} |X(w) f^{(2k+2)}(w)|^2$$
$$du dw dt$$

$$\leq \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t) \, |t|^{(2k+1)p} \, |t|^p \, ||f^{(2k+2)} X||_{L_p(-\infty,\infty)}^p \, dt$$

Thus,

$$(1.4.3) \quad R \leq \frac{M}{n^{(k+1)p}} \, ||f^{(2k+2)}||_{L_p(I_1)}^p$$

Hence from (1.4.1) and (1.4.3), we get

$$(1.4.4) \quad ||Z_2||_{L_p(I_2)} \leq \frac{M_1}{n^{k+1}} \, ||f^{(2k+2)}||_{L_p(I_1)} \, .$$

Combining (1.4.2) and (1.4.4), we get the result.

<u>Proof of theorem</u> 1.4.1. In view of lemma 1.3.5, it suffices to prove the theorem for $\beta \, \varepsilon \, B_\delta$ with $\delta < \min \, (a_2-x, y-b_2)$ where $a_1 < x < a_2 < b_2 < y < b_1$.

Now, for sufficiently small $\eta > 0$

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)}$$

$$\leq ||U_n(f-f_{\eta,2k+2},k,t)||_{L_p(I_2)}$$

$$+ ||U_n(f_{\eta,2k+2},k,t)-f_{\eta,2k+2}(t)||_{L_p(I_2)}$$

$$+ ||f_{\eta,2k+2}(t) - f(t)||_{L_p(I_2)}$$

$$(1.4.5) \quad = Z_1 + Z_2 + Z_3, \text{ say.}$$

Let X be the characteristic function of $[x,y]$. Then, from lemmas 1.3.7 and 1.3.6,

$$Z_1 \leq ||U_n(X(f-f_{\eta,2k+2}),k,t)||_{L_p(I_2)}$$

$$\leq ||f-f_{\eta,2k+2}||_{L_p[x,y]}$$

and hence by lemma 0.6.5

$$(1.4.6) \quad Z_1 \leq M_1\{\omega_{2k+2}(f,\eta,p,I_1)\}.$$

From lemma 1.4.2, we get

$$Z_2 \leq \frac{M_2}{n^{k+1}} \{||f_{\eta,2k+2}^{(2k+2)}||_{L_p[x,y]} + ||f_{\eta,2k+2}||_{L_p[x,y]}\},$$

which by lemma 0.6.5 implies

$$(1.4.7) \quad Z_2 \leq \frac{M_3}{n^{k+1}} \{\eta^{-(2k+2)}\omega_{2k+2}(f,\eta,p,I_1) + ||f||_{L_p(I_1)}\}.$$

Also, by the same lemma

$$(1.4.8) \quad Z_3 \leq M_4 \omega_{2k+2}(f,\eta,p,[x,y]).$$

Choosing η such that $\eta = n^{-1/2}$, from (1.4.6-8), we get

$$(1.4.9) \quad ||U_n(f,k,t) - f(t)||_{L_p(I_2)}$$

$$\leq M_6\{\omega_{2k+2}(f,\eta^{-1/2},p,I_1) + n^{-(k+1)}||f||_{L_p(I_1)}\}.$$

Since $\varphi \varepsilon \bar{\Phi}_{2k+2}$, $h^{2k+2} K_\varphi(h) \to 0$ as $h \to 0$ which implies that

$$t^{2k+2}/\varphi(t) \to 0 \text{ as } t \to C$$

and hence, from the hypothesis and (1.4.9), we get

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

which proves the theorem.

Corollary 1.4.3. Let $f \varepsilon L_p(-\infty,\infty)$ $(1 \le p < \infty)$, $\beta \varepsilon B^{\infty,2}$ and $\varphi \varepsilon \Phi_{2k+2}$. Then,

$$\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t)) \qquad (t \to 0)$$

implies that

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Proof.   Proceeding as in the proof of theorem 1.4.1, we get

$$(1.4.10) \quad ||U_n(f,k,t) - f(t)||_{L_p(I_2)}$$

$$\le M\{\omega_{2k+2}(f,n^{-1/2},p,I_1)+n^{-(k+1)}||f||_{L_p(I_1)}\}.$$

From (1.4.10) and the fact that $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$, we get that the left hand side of (1.4.10) is of $o(\varphi(n^{-1/2}))$ $(n \to \infty)$ and hence the corollary follows.

1.5 $O(\varphi)$-inverse theorem

In theorem 1.4.1, we proved that if $\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t))$ $(t \to 0)$ then

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} = O(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Now, we prove the corresponding local inverse theorem.

Theorem 1.5.1.   Let $\beta \varepsilon B^{\infty,2}$, $\varphi \varepsilon \Phi_{2k+2}$ and $f \varepsilon L_p(-\infty,\infty) (1\le p<\infty)$. Then,

$$(1.5.1) \quad ||U_n(f,k,t)-f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

(1.5.2) $\omega_{2k+2}(f,t,p,I_2) = O(\varphi(t))$ $(t \to 0)$.

Before proceeding to the proof of the theorem, we prove the following :

<u>Lemma</u> 1.5.2. Let $f \varepsilon L_p(-\infty,\infty)$ $(1 \leq p < \infty)$ and $\beta \varepsilon B_\delta$ with $\delta < \min (a_2-a_1,b_1-b_2)$. Then, for any $i,j \varepsilon \mathbb{N}^0$

$$||U_n((u-t)^i \int_t^u (u-w)^j f(w) \, dw, \, t)||_{L_p[a_2,b_2]}$$
$$\leq M \, n^{-(\frac{i+j+1}{2})} \, ||f||_{L_p[a_1,b_1]}.$$

<u>Proof</u> : In view of lemma 1.3.7, for all $t \varepsilon I_2$,

$$R = U_n((u-t)^i \int_t^u (u-w)^j f(w) \, dw, t)$$

$$= U_n(\chi(u)(u-t)^i \int_t^u (u-w)^j f(w) \, dw, t)$$

where $\chi(u)$ is the characteristic function of I_1. Then, for all $t \varepsilon I_2$,

$$|R| \leq | \, U_n(\chi(u) \, |u-t|^{i+j} \, |\int_t^u |f(w)|dw| \, ; t) |.$$

Now, proceeding as in the case of the estimate of R in lemma 1.4.2, we get that

$$||R||_{L_p(I_2)} \leq \frac{M}{n^{\frac{i+j+1}{2}}} \, ||f||_{L_p(I_1)}$$

and hence the result.

Lemma 1.5.3 : Let $\beta \in B_\delta$, for some $\delta > 0$ and $m \in \mathbb{N}$. Then, for $f \in L_p(-\infty,\infty)$ $(1 \le p < \infty)$ such that supp $f \subset (a_2,b_2)$,

$$||U_n^{(2m)}(f;t)||_{L_p(I_2)} = O(n^m||f||_{L_p(I_2)}) \quad (n \to \infty).$$

Further, if also, $f^{(2m-1)}$ exists and is absolutely continuous on $[a_2,b_2]$ and $f^{(2m)} \in L_p[a_2,b_2]$ then for all n sufficiently large

$$||U_n^{(2m)}(f;t)||_{L_p(I_2)} \le M||f^{(2m)}||_{L_p(I_2)} ,$$

M being a constant.

Proof :For $n > 2m$, it is clear that

$$U_n^{(2m)}(f;t) = \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} (\frac{\partial^{2m}}{\partial t^{2m}} \beta^n(t-u)) f(u) \, du.$$

Thus, by lemma 1.3.9, we get

$$U_n^{(2m)}(f;t) = \frac{1}{\alpha(n)} \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} n^{\nu+k} \int_{-\infty}^{\infty} \beta^{n-2m}(t-u)(\frac{\partial}{\partial t}\beta(t-u))^k \times$$
$$\times f(u) \, g_{\nu,k,m}(t-u) \, du$$

where $g_{\nu,k,m}$ are certain linear combinations of products of $\beta,\beta',\ldots,\beta^{(2m)}$ which are independent of n and hence bounded on $\mathbb{R}$.

Now, since $\beta'(0) = 0$, for some ξ lying between 0 and $t-u$, we have $\beta'(t-u) = (t-u)\beta''(\xi)$. Using this, we get

$$U_n^{(2m)}(f;t) = \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} n^{\nu+k} \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} (t-u)^k \beta^{n-2m}(t-u)(\beta''(\xi))^k \times$$
$$\times f(u) \, g_{\nu,k,m}(t-u) \cdot du$$

Using the boundedness of β'' and $g_{\nu,k,m}$ on $\mathbb{R}$,

$$|U_n^{(2m)}(f;t)| \leq M \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} n^{\nu+k} \frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^{n-2m}(t-u)|t-u|^k \times$$

$$\times |f(u)| \, du,$$

and hence, from corollary 1.3.2, for $f \in L_1(-\infty,\infty)$,

$$\int_{-\infty}^{\infty} |U_n^{(2m)}(f;t)| \, dt$$

$$\leq M_1 \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} n^{\nu+k} \frac{1}{\alpha(n-2m)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta^{n-2m}(u) |u-t|^k |f(t-u)| \, du \, dt$$

$$= M_1 \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} \frac{n^{\nu+k}}{\alpha(n-2m)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta^{n-2m}(u) |u-t|^k |f(t-u)| \, du \, dt$$

$$\leq M_2 \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} n^{\nu+k} \, n^{-k/2} \, ||f||_{L_1(-\infty,\infty)}$$

$$\leq M_3 \, n^m \, ||f||_{L_1(-\infty,\infty)}$$

and hence

$$||U_n^{(2m)}(f;t)||_{L_1(-\infty,\infty)} \leq M_3 \, n^m \, ||f||_{L_1(-\infty,\infty)} \, .$$

Similarly, we can prove that

$$||U_n^{(2m)}(f;t)||_{L_\infty(-\infty,\infty)} \leq M_4 \, n^m \, ||f||_{L_\infty(-\infty,\infty)} \, .$$

Hence, by (Riesz-Thorin's interpolation) lemma 0.6.4 , for $1 \leq p \leq \infty$

$$||U_n^{(2m)} f||_{L_p(-\infty,\infty)} \leq M_5 \, n^m \, ||f||_{L_p(-\infty,\infty)} \, ,$$

from which the first part of the lemma follows.

Since $f^{(m-1)}$ is absolutely continuous, $f^{(m)} \varepsilon L_p(-\infty,\infty)$ and $\beta \varepsilon B_\delta$, we can write

$$U_n^{(2m)}(f;t) = U_n(f^{(2m)};t),$$

from which the second part of the lemma is clear.

<u>Proof of the theorem 1.5.1</u> : Let (x_i,y_i), $i = 1,2,3,4,5$ satisfy

$$x_0 = a_1 < x_1 < x_2 < x_3 < x_4 < x_5 < a_2 < b_2 < y_5 < y_4 < y_3 < y_2$$

$< y_1 < b_1 = y_0$. Choose $g \varepsilon C_0^{2k+2}$ such that supp $g \subset (x_4,y_4)$ and

$g(t) = 1$ for $t \varepsilon [x_5,y_5]$. Let $\delta < \min(x_3-x_2,y_2-y_3)$. Then,

in view of lemma 1.3.5, it suffices to prove the theorem for

$\beta \varepsilon B_\delta$.

Writing $fg = \bar{f}$, we have, by applying Jensen's inequality

$2k+2$ times

$$\left| \int_o^\gamma \ldots \int_o^\gamma U_n^{(2k+2)}\left(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i\right) dz_1 \ldots dz_{2k+2} \right|^p$$

$$\leq \gamma^{(2k+2)(p-1)} \left\{ \int_o^\gamma \ldots \int_o^\gamma \left| U_n^{(2k+2)}\left(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i\right) \right|^p \right.$$

$$dz_1 \ldots dz_{2k+2} \; .$$

Hence, by Fubini's theorem, for all γ sufficiently small

$(\gamma < (2k+2)^{-1} \min \{x_3-x_2,y_2-y_3\})$

$$\int_{x_3}^{y_3} \left| \int_o^\gamma \ldots \int_o^\gamma U_n^{(2k+2)}\left(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i\right) dz_1 \ldots dz_{2k+2} \right|^p dt$$

$$\leq \gamma^{(2k+2)(p-1)} \int_o^\gamma \ldots \int_o^\gamma \int_{x_3}^{y_3} \left| U_n^{(2k+2)}\left(\bar{f},k,t+\sum_{i=1}^{2k+2} z_i\right) \right|^p$$

$$dt \; dz_1 \ldots dz_{2k+2}$$

$$\leq \gamma^{(2k+2)p} \, ||U_n^{(2k+2)}(\bar{f},k,t)||_{L_p[x_2,Y_2]}^p \, .$$

Thus, from lemma 0.6.1

$$||\Delta_\gamma^{2k+2} U_n(\bar{f},k,t)||_{L_p[x_3,Y_3]} \leq \gamma^{2k+2} \, ||U_n^{(2k+2)}(\bar{f},k,t)||_{L_p[x_2,Y_2]}$$

$$\leq \gamma^{2k+2} \{ ||U_n^{(2k+2)}(\bar{f}-\bar{f}_{\eta,2k+2},k,t)||_{L_p[x_2,Y_2]}$$

$$+ ||U_n^{(2k+2)}(\bar{f}_{\eta,2k+2},k,t)||_{L_p[x_2,Y_2]} \, .$$

Let η be sufficiently small ($\eta < \{4(k+1)\}^{-2} \times$

$$\times \min_{0 \leq i \leq 4} \{x_{i+1}-x_i, Y_i-Y_{i+1}\}) \, .$$

Then, using lemmas 1.5.3 and 0.6.5, it is easily seen that

$$(1.5.3) \quad ||\Delta_\gamma^{2k+2} U_n(\bar{f},k,t)||_{L_p[x_3,Y_3]}$$

$$= M_1 \, \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \, \omega_{2k+2}(\bar{f},\eta,p,[x_3,Y_3]) \, .$$

Next, we show that

$$(1.5.4) \quad ||\Delta_\gamma^{2k+2} \{U_n(\bar{f},k,t) - \bar{f}(t)\}||_{L_p[x_3,Y_3]} = O(\varphi(n^{-\frac{1}{2}}))(n \to \infty).$$

After having proved (1.5.4), combining with (1.5.3), we get

$$||\Delta_\gamma^{2k+2} \bar{f}(t)||_{L_p[x_3,Y_3]} \leq M_2 \{\varphi(n^{-1/2}) + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \times$$

$$\times \omega_{2k+2}(\bar{f},\eta,p,[x_3,Y_3])\} \, .$$

Choosing n such that $n \leq \eta^{-2} < 2n$, and taking supremum over $0 < \gamma \leq t$, for all t sufficiently small, we have

$$\omega_{2k+2}(\overline{f}, t, p, [x_3, y_3])$$

$$\leq M_3 \{\varphi(\eta) + (\frac{t}{\eta})^{2k+2} \omega_{2k+2}(\overline{f}, \eta, p, [x_3, y_3])\}$$

which by lemma 1.2.2 implies that

$$\omega_{2k+2}(\overline{f}, t, p, [x_3, y_3]) = O(\varphi(t)) \quad (t \to 0).$$

The assertion (1.5.2) follows from the fact that $\overline{f}(t) = f(t)$ for $t \varepsilon [a_2, b_2]$.

To prove (1.5.4) it suffices to prove that

$$(1.5.5) \quad ||U_n(\overline{f}, k, t) - \overline{f}(t)||_{L_p[x_3, y_3]} = O(\varphi(n^{-1/2})) \quad (n \to \infty).$$

We prove (1.5.5) by an induction as follows :

First, we prove the assertion (1.5.5) for all $\varphi \varepsilon \Phi_1$. Next, assuming the assertion (1.5.5) for all $\varphi \varepsilon \Phi_r$, $1 \leq r \leq 2k+1$, we prove it for $\varphi \varepsilon \Phi_{r+1}$.

Let $\varphi \varepsilon \Phi_1$. Then

$$||U_n(fg, k, t) - (fg)(t)||_{L_p[x_3, y_3]}$$

$$\leq ||U_n((f(u)-f(t)) g(t), k, t)||_{L_p[x_3, y_3]}$$

$$+ ||U_n(f(u)(g(u)-g(t)), k, t)||_{L_p[x_3, y_3]}$$

$$= ||g(t) \{U_n(f, k, t) - f(t)\}||_{L_p[x_3, y_3]} +$$

$$+ \ ||U_n(f(u)(u-t) \ g'(\xi),k,t)||_{L_p[x_3,Y_3]}$$

for some ξ lying between u and t

$$= T_1 + T_2, \text{ say.}$$

From hypothesis (1.5.1),

$$(1.5.6) \quad T_1 \leq M_4 \ \varphi(n^{-1/2}).$$

To estimate T_2, consider

$$R = ||U_n((u-t) \ f(u) \ g'(\xi);t)||^p_{L_p[x_3,Y_3]}$$

$$= \int_{x_3}^{Y_3} |\frac{1}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(t-u)(u-t) \ f(u) \ g'(\xi) \ du|^p \ dt$$

$$\leq \frac{1}{\alpha(n)} \int_{x_3}^{Y_3} \int_{-\infty}^{\infty} \beta^n(t-u) \ |u-t|^p \ |f(u)|^p \ |g'(\xi)|^p \ du \ dt,$$

by Jensen's inequality. Now, using Fubini's theorem and the boundedness of g',

$$R \leq \frac{M_5}{\alpha(n)} \int_{-\infty}^{\infty} \int_{x_3}^{Y_3} \beta^n(t-u) \ |u-t|^p \ |f(u)|^p \ dt \ du$$

$$= \frac{M_5}{\alpha(n)} \int_{-\infty}^{\infty} \beta^n(u) \ |u|^p \int_{x_3}^{Y_3} |f(t-u)|^p \ dt \ du$$

$$= \frac{M_5}{\alpha(n)} \int_{-\delta}^{\delta} \beta^n(u) \ |u|^p \int_{x_3}^{Y_3} |f(t-u)|^p \ dt \ du$$

$$\leq \frac{M_5}{\alpha(n)} \int_{-\delta}^{\delta} \beta^n(u) \ |u|^p \int_{a_1}^{b_1} |f(t)|^p \ dt \ du.$$

Hence, from corollary 1.3.4,

$$R \leq \frac{M_6}{n^{p/2}} ||f||^p_{L_p[a_1,b_1]}$$

which implies that

$$(1.5.7) \quad T_2 \leq \frac{M_7}{n^{1/2}} ||f||_{L_p[a_1,b_1]}.$$

Now, $\varphi \ \varepsilon \ \Phi_1$ implies that $hK_\varphi(h) \to 0$ as $h \to 0$, which in turn implies that

$$(1.5.8) \quad t/\varphi(t) = o(1 , \quad (t \to 0).$$

Combining (1.5.6-8), we get the assertion (1.5.5) for all $\varphi \ \varepsilon \ \Phi_1$. Next, assume the assertion (1.5.5) for all $\varphi \ \varepsilon \ \Phi_r$, $1 \leq r \leq 2k+1$. Let $\varphi \ \varepsilon \ \Phi_{r+1}$. Then

$$||U_n(fg,k,t) - (fg)(t)||$$

$$\leq ||U_n((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||U_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$\leq M_7 \ \varphi(n^{-1/2}) + ||U_n((f(u)-f_{\eta,2k+2}(u)) \times$$

$$\times (g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||U_n((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||U_n(f_{\eta,2k+2}(t)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$= M_8 \ \varphi(n^{-1/2}) \ Z_1 + Z_2 + Z_3, \quad \text{say.}$$

By lemmas 1.3.8 and lemma 0.6.5

$$(1.5.9) \qquad Z_3 \leq \frac{M_9}{n^{k+1}} \, ||f||_{L_p[x_2,y_2]}.$$

For some ξ lying between u and t

$$Z_1 = ||U_n((f(u)-f_{\eta,2k+2}(u))(u-t)\,g'(\xi),k,t)||_{L_p[x_3,y_3]}.$$

Proceeding as in the case of T_2 above, we get

$$(1.5.10) \qquad Z_1 \leq M_{10} \{n^{-1/2} \, ||f-f_{\eta,2k+2}||_{L_p[x_2,y_2]}\}.$$

For some ξ lying between u and t,

$$Z_2 \leq \frac{1}{(2k+1)!} \, ||U_n(\sum_{i=1}^{2k} \frac{g^{(i)}(t)}{i!} (u-t)^i \times$$

$$\times (\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w) \, dw),k,t)||_{L_p[x_3,y_3]}$$

$$+ \frac{1}{((2k+1)!)^2} \, ||U_n(g^{(2k+1)}(\xi)(u-t)^{2k+1} \times$$

$$\times (\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w) \, dw),k,t)||_{L_p[x_3,y_3]}$$

$$+ \sum_{i=1}^{2k+1} \sum_{j=1}^{2k} \frac{1}{i!j!} \, ||f_{\eta,2k+2}^{(i)}(t) g^{(j)}(t) \times$$

$$\times U_n((u-t)^{i+j},k,t)||_{L_p[x_3,y_3]}$$

$$+ \frac{1}{(2k+1)!} \sum_{i=1}^{2k+1} \frac{1}{i!} \, ||f_{\eta,2k+2}^{(i)}(t) \times$$

$$\times U_n((u-t)^{2k+1+i} g^{(2k+1)}(\xi),k,t)||_{L_p[x_3,y_3]}$$

$$= R_1 + R_2 + R_3 + R_4, \text{ say.}$$

From lemma 1.5.2, we get

$$(1.5.11) \quad R_1 \leq M_{11} \{ n^{-(k+3/2)} \, ||f_{\eta,2k+2}^{(2k+2)}||_{L_p[x_2,Y_2]} \}$$

and

$$(1.5.12) \quad R_2 \leq M_{12} \{ n^{-(2k+3/2)} \, ||f_{\eta,2k+2}^{(2k+2)}||_{L_p[x_2,Y_2]} \}.$$

Since $\sum\limits_{j=0}^{k} C(j,k) \, d_j^{-m} = 0$, $m = 1,2,\ldots,k$, from lemmas 1.3.3 and 0.6.3

$$(1.5.13) \quad R_3 \leq \frac{M_{13}}{n^{k+1}} \{ ||f_{\eta,2k+2}^{(2k+1)}||_{L_p[x_3,Y_3]} + || f_{\eta,2k+2}||_{L_p[x_3,Y_3]} \}$$

and

$$(1.5.14) \quad R_4 \leq \frac{M_{14}}{n^{k+1}} \{ ||f_{\eta,2k+2}^{(2k+1)}||_{L_p[x_3,Y_3]} + ||f_{\eta,2k+2}||_{L_p[x_3,Y_3]} \}$$

Thus, from (1.5.9-14) choosing n such that $n \leq \eta^{-2} < 2n$, it follows from lemma 0.6.5 , that

$$(1.5.15) \quad Z_1 + Z_2 + Z_3 \leq M_{15} \{ n^{-1/2} \, \omega_{2k+2}(f, n^{-1/2}, p, [x_1,Y_1])$$

$$+ n^{-1/2} \, \omega_{2k+1}(f, n^{-1/2}, p, [x_1,Y_1])$$

$$+ n^{-(k+1)} \, ||f||_{L_p[x_1,Y_1]} \}$$

Let $\varphi^*(t) = \frac{\varphi(t)}{t}$. Then

$$\frac{\varphi^*(t)}{\varphi^*(th)} = \frac{\varphi(t)}{t} \quad \frac{th}{\varphi(th)} = h \frac{\varphi(t)}{\varphi(th)}$$

and hence we may take $K_{\varphi^*}(h) = h \, K_{\varphi}(h)$.

which, since $\varphi \, \varepsilon \, \Phi_{r+1}$, implies that $\varphi^* \, \varepsilon \, \Phi_r$.

Hence, by induction hypothesis

$$||U_n(\bar{f},k,t) - \bar{f}(t)||_{L_p[x_3,y_3]} = O(\varphi^*(n^{-1/2}) \quad (n \to \infty).$$

This, in turn, implies (1.5.2), with φ replaced by φ^* and hence, since I_2 is arbitrary subject only to $I_2 \supset (a_1,b_1) = I_1^0$, we can conclude that, for the interval $[x_1,y_1]$,

(1.5.16) $\quad \omega_{2k+2}(f,t,p, [x_1,y_1]) = O(\varphi^*(t))\quad (t \to 0).$

Also, since $r \leq 2k+1$, $\varphi^* \, \varepsilon \, \Phi_{2k+1}$.

Thus, from (1.5.16) and lemma 1.2.4, we get

(1.5.17) $\quad \omega_{2k+1}(f,t,p, [x_1,y_1]) = O(\varphi^*(t))\quad (t \to 0).$

Now, from (1.5.15-17)

(1.5.18) $\quad Z_1+Z_2+Z_3 \leq M_{16}\{n^{-1/2}\, \varphi^*(n^{-1/2}) + n^{-1/2}\, \varphi^*(n^{-1/2})$

$$+ n^{-(k+1)}\, ||f||_{L_p[x_1,y_1]}\}.$$

Since $\varphi \, \varepsilon \, \Phi_{r+1} \subset \Phi_{2k+2}$,

(1.5.19) $\quad t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$.

Hence, from (1.5.18-19)

$$Z_1+Z_2+Z_3 \leq M_{17}\, \varphi(n^{-1/2})$$

and hence the theorem follows.

Corollary 1.5.4 : Let $1 \leq m \leq 2k+2$, $\varphi \in \Phi_m$, $\beta \in B^{\infty,2}$ and $f \in L_p(-\infty,\infty)$ $(1 \leq p < \infty)$. Then

$$||U_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})), \qquad (n \to \infty)$$

implies that

$$\omega_m(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Proof : Since $1 \leq m \leq 2k+2$ and $\varphi \in \Phi_m$, $\varphi \in \Phi_{2k+2}$. Hence, by theorem 1.5.1,

$$\omega_{2k+2}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Again, $\varphi \in \Phi_m$ implies that

$$h^m K_\varphi(h) \to 0 \qquad (h \to 0)$$

which implies, for $j = m, m-1, \ldots, 2k+1$, that

$$(1.5.20) \qquad h^j K_\varphi(h) \to 0 \qquad (h \to 0).$$

Now, lemma 1.2.2 and (1.5.20) for $j = 2k+1$, imply that

$$\omega_{2k+1}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0)$$

which by (1.5.20), for $j = 2k$, gives

$$\omega_{2k}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Continuing in this manner, for $j = m$, (1.5.20) implies that

$$\omega_m(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0)$$

which completes the proof.

1.6 o(φ)-inverse theorem

In corollary 1.4.3, we obtained that if

$$\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t)) \quad (t \to 0)$$

then

$$||U_n(f,k,t) - f(t)||_{L_p(I_2)} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Now, we prove the corresponding local o(φ)-inverse theorem.

Theorem 1.6.1.   Let $\beta \in B^{\infty,2}$, $\varphi \in \Phi_{2k+2}$ and $f \in L_p(-\infty,\infty)$ $(1 \le p < \infty)$. Then

$$||U_n(f,k,t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$(1.6.1) \quad \omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)) \quad (t \to 0).$$

Proof :   Let (x_i,y_i), $i = 1,2,3,4,5$ be such that $x_o = a_1 < x_1 < x_2 < x_3 < x_4 < x_5 < a_2 < b_2 < y_5 < y_4 < y_3 < y_2 < y_1 < b_1 = y_o$. In view of the lemma 1.3.5, we can assume that $\beta \in B_\delta$ with $\delta < \min(x_3-x_2, y_2-y_3)$. Choose $g \in C_o^{2k+2}$ such that supp $g \subset (x_4,y_4)$ and $g(t) = 1$ for $t \in [x_5,y_5]$. Writing $fg = \bar{f}$ and proceeding as in the proof of the theorem 1.5.1, for sufficiently small $\eta, \gamma > 0$, we get

$$(1.6.2) \quad ||\Delta_\gamma^{2k+2} U_n(\bar{f},k,t)||_{L_p[x_3,y_3]}$$

$$\le M_1 \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}})\omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3]).$$

Next, we shall prove that

$$(1.6.3) \qquad ||\Delta_\gamma^{2k+2}\{\bar{f}(t) - U_n(\bar{f}, k, t)\}||_{L_p[x_3, Y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

After having proved (1.6.3), we define $\psi(t)$ as follows :

$$\psi(t) = \begin{cases} ||\Delta_\gamma^{2k+2}\{\bar{f}(t) - U_n(\bar{f}, k, t)\}||_{L_p[x_3, Y_3]} & \text{if } t = n^{-1/2} \text{ for any } n \in \mathbb{N} \\ \psi((n^{-1/2})) & \text{if } t \in ((n+1))^{-1/2}, n^{-1/2}) \text{ for any } n \in \mathbb{N}. \end{cases}$$

Clearly

$$\psi(t) = o(\varphi(t)) \qquad (t \to 0).$$

Combining (1.6.2) and (1.6.3), we get

$$||\Delta_\gamma^{2k+2}\bar{f}(t)||_{L_p[x_3, Y_3]} \le M_2\{\psi(n^{-1/2})$$

$$+ \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}})\,\omega_{2k+2}(\bar{f}, \eta, p, [x_3, Y_3]))\}.$$

Choosing n such that $n \le \eta^{-2} < n+1$, for all sufficiently small t, it implies that

$$\omega_{2k+2}(\bar{f}, t, p, [x_3, Y_3]$$

$$\le M_3\{\psi(\eta) + (\frac{t}{\eta})^{2k+2}\,\omega_{2k+2}(\bar{f}, \eta, p, [x_2, Y_2]))\},$$

which, by lemma 1.2.3, gives

$$\omega_{2k+2}(\bar{f}, t, p, [x_3, Y_3]) = o(\varphi(t)) \qquad (t \to 0).$$

The conclusion (1.6.1), now, follows since $\bar{f}(t) = f(t)$ for $t \in I_2$.

We prove (1.6.3) by induction as in the theorem 1.5.1. That is, first we prove that the assertion (1.6.3) is true for all $\varphi \in \Phi_1$. Assuming that the assertion (1.6.3) is true for

$$(1.6.5) \quad \omega_{2k+2} \ (\bar{f}, t, p, [x_1, y_1]) = o(\varphi^*(t)) \quad (t \to \infty).$$

Since $\varphi^* \ \varepsilon \ \Phi_{2k+1}$, by lemma 1.2.5, we get

$$(1.6.6) \quad \omega_{2k+1} \ (\bar{f}, t, p, [x_1, y_1]) = o(\varphi^*(t) \quad (t \to 0).$$

Now, $\varphi \ \varepsilon \ \Phi_{r+1} \subset \Phi_{2k+2}$, implies that

$$h^{2k+2} \ K_{\varphi}(h) \to 0 \quad as \ h \to 0$$

which implies that

$$(1.6.7) \quad t^{2k+2}/\varphi(t) \to 0 \quad as \ t \to 0.$$

Combining $(1.6.4-7)$, we get

$$Z_1 + Z_2 + Z_3 = o(\varphi(n^{-1/2}))$$

which proves $(1.6.3)$ and hence the theorem.

<u>Corollary</u> 1.6.2 : Let $\beta \ \varepsilon \ B^{\infty, 2}$, $\varphi \ \varepsilon \ \Phi_m$, $1 \leq m \leq 2k+2$ and $f \ \varepsilon \ L_p(-\infty, \infty)$ $(1 \leq p < \infty)$. Then

$$||U_n(f, k, t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_m(f, t, p, I_2) = o(\varphi(t)) \quad (t \to 0).$$

<u>Proof</u> : Since $\varphi \ \varepsilon \ \Phi_m \subset \Phi_{2k+2}$, theorem 1.6.1 implies that

$$\omega_{2k+2} \ (f, t, p, I_2) = o(\varphi(t)) \quad (t \to 0).$$

Again, $\varphi \in \Phi_m$, implies that

$$h^m K_\varphi(h) \to 0 \qquad (h \to 0)$$

which, in turn, implies that, for $j = m, m+1, \ldots, 2k+1$,

$$(1.6.8) \qquad h^j K_\varphi(h) \to 0 \qquad (h \to 0).$$

Taking $j = 2k+1$, in (1.6.8), from lemma 1.2.5, we get

$$\omega_{2k+1}(f, t, p, I_2) = o(\varphi(t)) \qquad (t \to 0)$$

which, again by (1.6.8) for $j = 2k$, gives

$$\omega_{2k}(f, t, p, I_2) = o(\varphi(t)) \qquad (t \to 0).$$

Continuing in this manner, for $j = m$, (1.6.8) implies that

$$\omega_m(f, t, p, I_2) = o(\varphi(t)) \qquad (t \to 0).$$

CHAPTER II

φ-INVERSE THEOREMS FOR LINEAR COMBINATIONS AND INTERPOLATORY MODIFICATIONS OF BERNSTEIN-KANTOROVITCH POLYNOMIALS

2.1   Introduction

In this chapter we study the linear combinations $P_n(\cdot, k, t)$ and interpolatory modifications $P_{n,m}(\cdot, t)$ of the Bernstein-Kantorovitch polynomials P_n which themselves are a modification of Bernstein polynomials suggested by Kantorovitch [29] for approximation of functions belonging to $L_p[0,1]$, given by

$$P_n(f, t) = (n+1) \left\{ \sum_{\nu=0}^{n} p_{n\nu}(t) \int_{\frac{\nu}{(n+1)}}^{\frac{(\nu+1)}{(n+1)}} f(u)\, du \right\}$$

where

(2.1.1) $p_{n\nu}(t) = \binom{n}{\nu} t^{\nu}(1-t)^{n-\nu}$.

For any $k+1$ distinct positive integers $d_0, \ldots, d_k$, the linear combinations $P_n(\cdot, k, t)$ of $P_n(\cdot, t)$ are defined by

$$P_n(\cdot, k, t) = \sum_{j=0}^{k} C(j, k)\, P_{d_j n}(\cdot, t), \quad n \in \mathbb{N}$$

where

$$C(j, k) = \begin{cases} \prod_{\substack{i=0 \\ i \neq j}}^{k} \dfrac{d_j}{d_j - d_i} & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases}$$

The interpolatory modifications, $P_{n,m}(\cdot, t)$, introduced by Sinha [61], of the Bernstein-Kantorovitch polynomials $P_n(\cdot, t)$, are modifications of $P_n(\cdot, t)$ by making use of the classical Newton's interpolation polynomials. This is accomplished by replacing function value $f(u)$ at the points 'u' by Newton's interpolation polynomial of mth degree based at the nodes $u, u + \frac{1}{n^{1/2}}, \ldots, u + \frac{m}{n^{1/2}}$.

Thus, for any $f \in L_p[0,1]$ $(1 \le p < \infty)$, interpolatory modifications $P_{n,m}(f; t)$ of order m, of the Bernstein-Kantorovitch polynomials $P_n(f; t)$ are defined as follows:

$$P_{n,m}(f; t) = (n+1) \sum_{\nu=0}^{n} \sum_{j=0}^{m} \frac{n^{j/2}}{j!} p_{n\nu}(t) \int_{\frac{\nu}{n+1}}^{\frac{\nu+1}{n+1}} \left(\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}}) \right) \times$$
$$\times \Delta^j f(u) \, du,$$

where $\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})$ for $j=0$ is interpreted as 1 and $f(u)$ is regarded as zero when $u > 1$. Here and in the sequel Δ denotes $\Delta_{n^{-1/2}}$.

Approximation by Bernstein-Kantorovitch polynomials in L_p-norms $(1 \le p < \infty)$ has been studied by Lorentz [34], Butzer [15], Hoeffding [26], Bojanic and Shisha [13], Ditzian and May [22], Grundmann [24], Muller [44], Maier [37,33], Riemenschneider [54], May [40], Becker and Nessel [7,8] and recently by Winslin [67]. These works establish that the optimal rate of convergence for the Bernstein-Kantorovitch polynomials in L_p-norms is $O(n^{-1})$,

which is also the optimal rate of convergence for the corresponding original operators, viz., the Bernstein polynomials with respect to sup-norm.

Ditzian and May [22] obtained local direct, inverse and saturation theorems in L_p-norms in the set up of contracting subintervals. Grundmann [24] and Muller [44] obtained bounds for the error in L_p-approximation for the cases p=1 and p>1, respectively, in terms of the first order integral modulus of smoothness of the functions. Similar estimates were obtained in weighted L_1-norm by Bojanic and Shisha [13]. Bounds for the error in the approximation of differentiable functions have been obtained by Muller [44] and Hoeffding [26] respectively.

Maier [37] proved the global saturation theorem for $P_n(\cdot,t)$ in $L_1[0,1]$-norm, which has been extended to $L_p[0,1](1 < p < \infty)$ by Riemenschineider [54]. Becker and Nessel [7] characterised the saturation class of Bernstein-Kantorovitch polynomials and May [40] proved a modified global saturation theorem alongwith a correction term in a weighted $L_p[0,1]$-norm $(1 \leq p < \infty)$.

Sinha [61] obtained $O(n^{-\alpha/2})$ local direct, inverse and saturation theorems for linear combinations and interpolatory modifications of Bernstein-Kantorovitch polynomials in L_p-norm $(1 \leq p < \infty)$ over contracting subintervals. Recently, Winslin [67] has obtained

global direct, inverse theorems for linear combinations of Bernstein-Kanotorovitch polynomials, in L_p-norm $(1 \leq p < \infty)$, for the order $O(n^{-\alpha/2})$.

In the present chapter, we extend the results of Sinha [61] to a more general order $O(\varphi(n^{-1/2}))$ for linear combinations and interpolatory modifications of Bernstein-Kantorovitch polynomials in L_p-norm $(1 \leq p < \infty)$. These results are also local in nature over contracting subintervals. In this chapter, section 2.2 contains some basic results about the operators $P_n(\cdot,t)$, $P_{n,m}(\cdot,t)$ and $P_n(\cdot,k,t)$ which shall be used in the later sections. In section 2.3 we obtain $O(\varphi)$-inverse theorem for $P_n(\cdot,k,t)$ and section 2.4 consists of $o(\varphi)$-inverse theorem for $P_n(\cdot,k,t)$. Similarly, sections 2.5 and 2.6, respectively contain $O(\varphi)$- and $o(\varphi)$-inverse theorems for the operator $P_{n,m}(\cdot,t)$.

2.2   Basic Results

Throughout the rest of this chapter $I = [0,1]$, $I_j = [a_j,b_j]$, $j=1,2,3$ where $0 < a_j < a_{j+1}$ and $b_{j+1} < b_j < 1$.

Lemma 2.2.1 [35] : Let $s \in \mathbb{N}$. Then $T_{n,s}(t)$ defined as

$$T_{n,s}(t) = \sum_{\nu=0}^{n} (\nu-nt)^s p_{n\nu}(t)$$

is a polynomial in t and n; in t of degree $\leq s$, in n of degree $[s/2]$, where $[\cdot]$ stands for the integral part and $p_{n\nu}(t)$ is given by (2.1.1), $T_{n,2s}(t)$ depends only

on $t(1-t)$ and n with the coefficient of n^s being $\frac{(2s)!}{2^s s!} (t(1-t))^s$. $T_{n,2s+1}(t)$ is a polynomial in $t(1-t)$ and n multiplied by the factor $(1-2t)$.

Lemma 2.2.2: Let r be a positive number. Then, for $t \varepsilon [0,1]$,

$$|T_{n,r}(t)| \leq Mn^{r/2};$$ M being a constant independent of t and n.

May [40] expressed the moments of Bernstein-Kantorovitch polynomials in terms of the moments of the Bernstein polynomials as follows:

Lemma 2.2.3: Let $m \varepsilon \mathbb{N}$ and $p(t) = t(1-t)$. Then

$$P_n((u-t)^m ; t) = \frac{n+1}{(m+1) \ p(t)} B_{n+1} ((u-t)^{m+2}; t)$$

where

$$B_n (f; t) = \sum_{\nu=0}^{n} p_{n\nu}(t) \ f(\nu/n)$$

and

$$p_{n\nu}(t) \text{ is given by } (2.1.1).$$

From lemmas 2.2.1 and 2.2.3, we get

Corollary 2.2.4: For $m \varepsilon \mathbb{N}$, there holds

$$P_n((u-t)^m ; t) = \frac{1}{(n+1)^{m+1}} Q(n+1; t)$$

where $Q(n+1; t)$ is a polynomial in $(n+1)$ of degree $[\frac{m+2}{2}]$ and in t of degree $\leq m$.

Corollary 2.2.5: Let $r \in \mathbb{N}$. Then, for $t \in I$,

$$P_n(|u-t|^r \; ; t) \leq \frac{M}{n^{r/2}} \; ,$$

M being a constant independent of t and n.

Lemma 2.2.6 ([22]): Let $f \in L_p[0,1]$ $(1 \leq p < \infty)$. Then

$$||P_n(f;t)||_{L_p[0,1]} \leq ||f||_{L_p[0,1]} \; .$$

Sinha [61] introduced the duals P_n^* of the operators P_n as follows:

Definition 2.2.7: The dual operator sequence $\{P_n^*\}$ of $\{P_n\}$ is defined as

$$P_n^*(f;u) = \int_0^1 K(n,t,u) \, f(t) \, dt$$

where $K(n,t,u)$ is given by

$$(2.2.1) \quad \left\{ \begin{array}{l} K(n,t,u) = (n+1)\{\sum_{\nu=0}^{n} p_{n\nu}(t) \, X_{n\nu}(u)\}, \\ X_{n\nu}(u) \text{ is the characteristic function of} \\ \text{the interval } [\frac{\nu}{n+1} , \frac{\nu+1}{n+1}) \text{ for } \nu = 0,1,\ldots,n-1 \\ \text{and of } [\frac{n}{n+1} , 1] \text{ for } \nu = n. \end{array} \right.$$

The following lemma of Sinha [61] gives the order of the moments of the dual operator P_n^*.

Lemma 2.2.8: Let $k \in \mathbb{N}$ and $u \in I_1$. Then

$$P_n^*(|u-t|^k;u) = O(n^{-k/2}) \quad (n \to \infty)$$

uniformly in $u \in I_1$.

Ditzian and May[22] proved the following

Lemma 2.2.9: Let $f \in L_p[0,1]$ $(1 \le p < \infty)$, $i \in \mathbb{N}^0$, $[a_1, b_1] \subset (a,b)$ and $X(u)$ denote the characteristic function of $[a,b]$. Then, for any fixed positive number ℓ,

$$|| P_n(f(u)(u-t)^i(1-X(u)); t) ||_{L_p[a_1,b_1]}$$
$$= O(n^{-\ell}) \, || f ||_{L_p[0,1]}$$

The proof of the above result in [22] seems somewhat ambiguous. An alternative proof of the same may be given as follows:

With $K(n,t,u)$ as in (2.2.1), $\ell > 0$ and $0 < \delta < \min(a_1-a, b-b_1)$, using Jensen's inequality, we have

$$|| P_n(f(u)(u-t)^i(1-X(u)); t) ||_{L_p[a_1,b_1]}^p$$

$$= \int_{a_1}^{b_1} | \int_0^1 K(n,t,u) \, f(u)(u-t)^i(1-X(u)) du |^p dt$$

$$\le \int_{a_1}^{b_1} \int_0^1 K(n,t,u) \, |f(u)|^p |u-t|^{ip}(1-X(u)) du \, dt$$

$$\le \int_{a_1}^{b_1} \int_0^1 K(n,t,u) \, |f(u)|^p \frac{|u-t|^{ip+\ell}}{\delta^\ell}(1-X(u)) du \, dt$$

$$\le \int |f(u)|^p (\int_0^1 K(n,t,u) \frac{|u-t|^{ip+\ell}}{\delta^\ell} dt) du$$

$$\le \frac{M}{n^{(ip+\ell)/2}} || f ||_{L_p[0,1]}^p \, ,$$

by corollary 2.2.5. Since ℓ is arbitrary, we get the result.

The moment estimates of the operators $P_{n,m}(\cdot,t)$ were obtained by Sinha [61] and are given in

Lemma 2.2.10   Let $k \in \mathbb{N}$.   Then, denoting $p(t) = t(1-t)$, we have

(i)   If $k \leq m$, $P_{n,m}((u-t)^k;t) = 0$; $t \in I$

(ii) $P_{n,m}((u-t)^{m+1};t) = (-1)^m P_n(\prod_{i=0}^{m} (u-t+ \frac{i}{n}_{1/2});t)$

$$= (-1)^m \frac{n+1}{p(t)} \{ \sum_{r=0}^{m} a_r B_{n+1}((u-t)^{m-r+3};t) \}$$

where a_r's are certain positive constants.

(iii)   If $k > m+1$

$$P_{n,m}((u-t)^k;t) = \frac{n+1}{p(t)} \{ \sum_{r=0}^{k-1} \frac{b_r}{n^r/2} B_{n+1}((u-t)^{k-r+2};t) \}$$

where b_r's are certain constants.

The following theorem of Sinha [61] describes the convergence of the interpolatory modifications:

Theorem 2.2.11   Let $f \in L_p(I)$ $(1 \leq p < \infty)$.   Then

$$\lim_{n \to \infty} P_{n,m} f = f ,$$

the convergence being in the space $L_p(I)$.

The following two theorems of Sinha [61] give an estimate for the error in the approximation of functions in $C^{2k+2}(I)$ and $C^{m+1}(I)$, respectively, for the operators $P_n(\cdot,k,t)$ and $P_{n,m}(\cdot,t)$.

Theorem 2.2.12   Let $f \varepsilon C^{2k+2}(I)$.   Then there holds

$$P_n(f,k,t) - f(t) = n^{-(k+1)} \sum_{i=1}^{2k+2} Q(i,k,t) \, f^{(i)}(t)$$

$$+ o(n^{-(k+1)}) \quad (n \to \infty)$$

uniformly in $t \varepsilon I$, where $Q(i,k,t)$ are certain polynomials in t.

Theorem 2.2.13   Let $f \varepsilon C^{m+1}(I)$.   Then

$$P_{n,m}(f;t) - f(t) = \frac{(-1)^m}{(m+1)!} \frac{p_{m+1}(t)}{n^{(m+1)/2}} f^{(m+1)}(t) + o(\frac{1}{n^{(m+1)/2}}) (n \to \infty)$$

and

$$P_{n,m+1}(f,t) - f(t) = o(\frac{1}{n^{(m+1)/2}}) \quad (n \to \infty)$$

uniformly in $t \varepsilon I$, where $p_{m+1}(t)$ is a polynomial in t of degree $\leq m+1$ and $p_{m+1}(t) > 0$ for t in the interior of I.

The following lemma of Sinha [61] can be characterised as Bernstein type inequalities for the operators $P_n(\cdot,t)$ and $P_{n,m}(\cdot,t)$.

Lemma 2.2.14   Let $h \varepsilon L_p(I)$ $(1 \leq p < \infty)$ have supp h $\subset I_2$. Then

(1) $|| P_n^{(m+1)}(h;t) ||_{L_p(I_2)} \leq M_1 n^{(\frac{m+1}{2})} ||h||_{L_p(I_2)}$

and

(2) $|| P_{n,m}^{(m+1)}(h;t) ||_{L_p(I_2)} \leq M_3 n^{(\frac{m+1}{2})} ||h||_{L_p(I_2)}$

If, in addition, h has $m+1$ derivatives with $h^{(m)} \varepsilon$ A.C (I_2) and $h^{(m+1)} \varepsilon L_p(I_2)$ then

$$(3) \qquad ||P_n^{(m+1)}(h;t)||_{L_p(I_2)} \le M_2 ||h^{(m+1)}||_{L_p(I_2)} \, .$$

and

$$(4) \qquad ||P_{r,m}^{(m+1)}||_{L_p(I_2)} \le M_4 ||h^{(m+1)}||_{L_p(I_2)} \, .$$

Here, M_i, $i=1,2,3,4$ are constants independent of n and h.

Next, we give some technical lemmas of Sinha [61] which are of use in the proofs of our inverse theorems.

<u>Lemma 2.2.15</u> Let $h \varepsilon L_p(I)$ $(1 \le p < \infty)$ and $i,j \varepsilon \mathbb{N}^o$. Then for any fixed $\ell > 0$, there holds

$$||(n+1) \sum_{\nu=0}^{n} \{P_{n\nu}(t) |\frac{\nu}{n} - t|^i {}_{\frac{\nu}{(n+1)}}^{\frac{(\nu+1)}{n+1}} |u-t|^j | \int_t^u |h(w)| dw| du\}||_{L_p(I_2)}$$

$$\le M\{n^{-(i+j+1)/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p(I)}\},$$

where M is a constant independent of n and h.

<u>Lemma 2.2.16</u> Let $i \varepsilon \mathbb{N}$ and $h \varepsilon L_p(I)$ $(1 \le p < \infty)$. Then, for any fixed $\ell > 0$, there holds

$$||P_n(|u-t|^i|h(u)|;t)||_{L_p(I_2)}$$

$$\le M\{\frac{1}{n^{i/2}}||h||_{L_p(I_1)} + \frac{1}{n^{\ell}}||h||_{L_p(I)}\},$$

M being independent of n and h.

Lemma 2.2.17   Let $j, k, s \in \mathbb{N}^o$, $1 \le p < \infty$ and $h \in L_p(I)$.

Then for any fixed $\ell > 0$ and for all sufficiently large values of n

$$\left\| (n+1) \left\{ \sum_{\nu=0}^{n} p_{n\nu}(t) \left| \frac{\nu}{n} - t \right|^j \left\{ \frac{(\nu+1)}{(n+1)} \atop \frac{\nu}{(n+1)} \right\} |u-t|^k \times \right. \right.$$

$$\left. \left. \times \left| \int_{t}^{u+\frac{s}{n^{1/2}}} |h(w)| \, dw \right| du \right\} \right\|_{L_p(I_2)}$$

$$\le M\{n^{-(k+j+1)/2} \|h\|_{L_p(I_1)} + n^{-\ell} \|h\|_{L_p(I)}\}$$

where M does not depend on n and h.

We combine the local direct theorems of Sinha [61] for the operators $P_n(\bullet, k, t)$ and $P_{n,m}(\bullet, t)$ in

Theorem 2.2.18   Let $f \in L_p(I)$ $(1 \le p < \infty)$.   Then, for all sufficiently large values of n

(1) $\|P_n(f, k, t) - f(t)\|_{L_p(I_2)} \le M_1 \{\omega_{2k+2}(f, n^{-1/2}, p, I_1)$

$$+ n^{-(k+1)} \|f\|_{L_p(I)}\}$$

and

(2) $\|P_{n,m}(f, t) - f(t)\|_{L_p(I_2)}$

$$\le M_2 \{\omega_{m+1}(f, n^{-1/2}, p, I_1) + n^{-(m+1)} \|f\|_{L_p(I)}\},$$

M_1, M_2 being constants.

We conclude this section by stating the local $O(n^{-\alpha/2})$-inverse theorems of Sinha [61] for the operators $P_n(\bullet, k, t)$ and $P_{n,m}(\bullet, t)$.

<u>Theorem</u> 2.2.19 Let $0 < \alpha < 2k+2$ and $f \varepsilon L_p(I)$ $(1 \le p < \infty)$.
Then

$$||P_n(f,k,t) - f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}) \qquad (n \to \infty)$$

implies that

$$\omega_{2k+2}(f,t,p,I_2) = O(t^{\alpha}) \qquad (t \to 0).$$

<u>Theorem</u> 2.2.20   Let $0 < \alpha < m+1$ and $f \varepsilon L_p(I)$ $(1 \le p < \infty)$.
Then

$$||P_{n,m}(f,t) - f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}) \qquad (n \to \infty)$$

implies that

$$\omega_{m+1}(f,t,p,I_2) = O(t^{\alpha}) \qquad (t \to 0).$$

Theorems 2.2.19-20 will follow as very special cases
of our $O(\varphi)$-inverse theorems 2.3.1 and 2.5.1.

2.3 $O(\varphi)$-inverse theorem for $P_n(\cdot,k,t)$

By theorem 2.2.18, if $\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t))$ $(t \to 0)$,
then $||P_n(f,k,t) - f(t)||_{L_p(I_2)} = O(\varphi(n^{-1/2}))$, $(n \to \infty)$. The
corresponding inverse theorem is as follows:

<u>Theorem</u> 2.3.1 Let $\varphi \varepsilon \Phi_{2k+2}$ and $f \varepsilon L_p[0,1]$ $(1 \le p < \infty)$.
Then

$$(2.3.1) \qquad ||P_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$(2.3.2) \qquad \omega_{2k+2}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Proof: Let x_i, y_i's, g and $\bar{f}$ be defined as in the proof of Theorem 1.5.1.   Then

$$|| \Delta_\gamma^{2k+2} \bar{f}(t) ||_{L_p[x_3, y_3]} \leq || \Delta_\gamma^{2k+2} \{\bar{f}(t) - P_n(\bar{f}, k, t)\} ||_{L_p[x_3, y_3]}$$

$$+ || \Delta_\gamma^{2k+2} P_n(\bar{f}, k, t) ||_{L_p[x_3, y_3]} .$$

By 2k+2 applications of Jensen's inequality

$$|\int_o^\gamma \dots \int_o^\gamma P_n^{(2k+2)} (\bar{f}, k, t + \sum_{i=1}^{2k+2} z_i) dz_1 \dots dz_{2k+2}|^p$$

$$\leq \gamma^{(2k+2)(p-1)} \{\int_o^\gamma \dots \int_o^\gamma |P_n^{(2k+2)} (\bar{f}, k, t + \sum_{i=1}^{2k+2} z_i)|^p$$

$$dz_1 \dots dz_{2k+2} \}.$$

Now, using Fubini's theorem, for all γ sufficiently small,

$$\int_{x_3}^{y_3} |\int_o^\gamma \dots \int_o^\gamma P_n^{(2k+2)} (\bar{f}, k, t + \sum_{i=1}^{2k+2} z_i) dz_1 \dots dz_{2k+2}|^p dt$$

$$\leq \gamma^{(2k+2)p} || P_n^{(2k+2)} (\bar{f}, k, t) ||_{L_p[x_3', y_3']}^p ,$$

where $x_3' = x_3$ and $y_3' = y_3 + (2k+2)\nu$.

Thus, by lemma 0.6.1   , for all η sufficiently small (as in the proof of the theorem 1.5.1.).

$$|| \Delta_\gamma^{2k+2} P_n(\bar{f}, k, t) ||_{L_p[x_3, y_3]} \leq \gamma^{2k+2} || P_n^{(2k+2)} (\bar{f}, k, t) ||_{L_p[x_3', y_3']}$$

$$\leq \gamma^{2k+2} \{ || P_n^{(2k+2)} (\bar{f} - \bar{f}_{\eta, 2k+2}, k, t)) ||_{L_p[x_3', y_3']} +$$

$$+ \; ||P_n^{(2k+2)}(\bar{f},2k+2,k,t)||_{L_p}[x_3',Y_3']\}$$

$$\leq M_1 \; \gamma^{2k+2}\{n^{k+1}||\bar{f}-\bar{f}_{\eta,2k+2}||_{L_p}[x_3',Y_3']$$

$$+ \; ||\bar{f}_{\eta,2k+2}||_{L_p}[x_3',Y_3']\} \; ,$$

by parts 1 and 3 of lemma 2.2.14. Now, applying lemma 0.6.5

$$(2.3.3) \quad ||\Delta_\gamma^{2k+2}P_n(\bar{f},k,t)||_{L_p}[x_3,Y_3]$$

$$\leq M_2\gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \; \omega_{2k+2}(\bar{f},\eta,p,[x_3,Y_3]).$$

The next major step is to show that

$$(2.3.4) \quad ||\Delta_\gamma^{2k+2}\{\bar{f}(t)-P_n(\bar{f},k,t)\}||_{L_p}[x_3,Y_3] = O(\varphi(n^{-1/2})) \; (n\to\infty).$$

For, after having proved (2.3.4), from (2.3.3), we get

$$||\Delta_\gamma^{2k+2}\bar{f}(t)||_{L_p}[x_3,Y_3] \leq M_3\{\varphi(n^{-1/2})$$

$$+ \; \gamma^{2k+2}(n^{k+1}+\frac{1}{\eta^{2k+2}}) \; \omega_{2k+2}(\bar{f},\eta,p,[x_3,Y_3]) \; ,$$

Choosing n such that $n \leq \eta^{-2} < 2n$ and taking
supremum over $0 < \gamma \leq t$, for all sufficiently small t, we have

$$\omega_{2k+2}(\bar{f},t,p,[x_3,Y_3]) \leq M_4\{\varphi(\eta)+(t/\eta)^{2k+2}\omega_{2k+2}(\bar{f},\eta,p,[x_3,Y_3])$$

which by lemma 1.2.2, gives

$$\omega_{2k+2}(\bar{f},t,p,[x_3,Y_3]) = O(\varphi(t)) \qquad (t \to 0)$$

and the conclusion follows since $\bar{f}(t) = f(t)$ on I_2.

To prove (2.3.4), it suffices to prove that

(2.3.5) $\quad ||P_n(\bar{f},k,t) - \bar{f}(t)||_{L_p[x_3,y_3]}= O(\varphi(n^{-1/2}))$ $\quad$ (n → ∞).

We prove this by an inductive argument as in the case of Theorem 1.5.1.

$\quad$ First, we prove the theorem for all $\varphi \in \Phi_1$. Next, assuming that for some $r \in \mathbb{N}$, such that $1 \leq r \leq 2k+1$, the theorem holds, for all $\varphi \in \Phi_r$, we prove the theorem for all $\varphi \in \Phi_{r+1}$.

Hence let $\varphi \in \Phi_1$. Then

$$||P_n(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq ||P_n((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||P_n(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$= ||g(t)\{P_n(f,k,t)-f(t)\}||_{L_p[x_3,y_3]}$$

$$+ ||P_n(f(u)(u-t)g'(\xi),k,t)||_{L_p[x_3,y_3]}$$

for some ξ lying between u and t.

Now, the hypothesis (2.3.1) alongwith lemma 2.2.16, implies that,

(2.3.6) $\quad ||P_n(fg,k,t)-(fg)(t)||_{L_p[x_3,y_3]} \leq M_5 \varphi(n^{-1/2})+M_6 n^{-1/2}$.

Since, $\varphi \in \Phi_1$, $hK_\varphi(h) \to 0$ as $h \to 0$. Hence,

$$(2.3.7) \qquad t/\varphi(t) \to 0 \quad \text{as} \quad t \to 0.$$

Combining (2.3.6-7), we get (2.3.5), for all $\varphi \in \Phi_1$.
Hence, the theorem is proved for all $\varphi \in \Phi_1$. Next assume
that for some $r \in \mathbb{N}$ such that $1 \leq r \leq 2k+1$, the theorem
holds, for all $\varphi \in \Phi_r$. Let $\varphi \in \Phi_{r+1}$. Then,

$$||P_n(fg, k, t) - (fg)(t)||_{L_p[x_3, y_3]}$$

$$\leq M_6' \varphi(n^{-1/2}) + ||P_n(f(u)(g(u) - g(t)), k, t)||_{L_p[x_3, y_3]}$$

$$\leq M_6' \varphi(n^{-1/2}) + ||P_n((f(u) - f_{\eta, 2k+2}(u)) \times$$

$$\times (g(u) - g(t)), k, t)||_{L_p[x_3, y_3]}$$

$$+ ||P_n((f_{\eta, 2k+2}(u) - f_{\eta, 2k+2}(t))(g(u) - g(t)), k, t)||_{L_p[x_3, y_3]}$$

$$+ ||P_n(f_{\eta, 2k+2}(t)(g(u) - g(t)), k, t)||_{L_p[x_3, y_3]}$$

$$= M_6' \varphi(n^{-1/2}) + z_1 + z_2 + z_3, \text{ say}.$$

We have, for some ξ lying between u and t,

$$z_1 \leq ||P_n((f(u) - f_{\eta, 2k+2}(u))(u - t)g'(\xi), k, t)||_{L_p[x_3, y_3]}$$

$$\leq ||g'||_{C(I)} \{ \sum_{j=0}^{k} |C(j, k)| \times$$

$$\{ ||P_{d_j n}(|f(u) - f_{\eta, 2k+2}(u)| |u - t|, t)||_{L_p[x_3, y_3]}$$

Hence, using lemma 2.2.16,

$$(2.3.8) \qquad z_1 \leq M_7 \{ n^{-1/2} ||f - f_{\eta, 2k+2}||_{L_p[x_2, y_2]} + n^{-(k+1)} ||f||_{L_p(I)} \}$$

Again, for some ξ lying between u and t

$$(f_{\eta,2k+2}(u) - f_{\eta,2k+2}(t))(g(u) - g(t))$$

$$= \left\{ \sum_{i=1}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}_{\eta,2k+2}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f^{(2k+2)}_{\eta,2k+2}(w)\,dw \right\} \times$$

$$\times \left\{ \sum_{i=1}^{2k} \frac{(u-t)^i}{i!} g^{(i)}(t) + \frac{(u-t)^{2k+1}}{(2k+1)!} g^{(2k+1)}(\xi) \right\}$$

$$= \sum_{i=1}^{2k+1} \sum_{j=1}^{2k} \frac{f^{(i)}_{\eta,2k+2}(t)}{i!\,j!} g^{(j)}(t)(u-t)^{i+j}$$

$$+ \frac{g^{(2k+1)}(\xi)}{(2k+1)!} \left\{ \sum_{i=1}^{2k+1} (u-t)^{2k+1+i} f^{(i)}_{\eta,2k+2}(t) \right\}$$

$$+ \frac{1}{(2k+1)!} \left\{ \sum_{i=1}^{2k} \frac{g^{(i)}(t)}{i!} (u-t)^i \int_t^u (u-w)^{2k+1} f^{(2k+2)}_{\eta,2k+2}(w)\,dw \right\}$$

$$+ \frac{1}{((2k+1)!)^2} \left\{ g^{(2k+1)}(\xi)(u-t)^{2k+1} \int_t^u (u-w)^{2k+1} f^{(2k+2)}_{\eta,2k+2}(w)\,dw \right\}$$

Hence

$$(2.3.9) \qquad Z_2 \leq R_1 + R_2 + R_3 + R_4, \quad \text{where}$$

$$R_1 = \sum_{i=1}^{2k+1} \sum_{j=1}^{2k} \frac{1}{i!\,j!} ||f^{(i)}_{\eta,2k+2}(t) g^{(j)}(t)$$

$$P_n((u-t)^{i+j}, k, t)||_{L_p[x_3, Y_3]},$$

$$R_2 = \frac{1}{(2k+1)!} \left\{ \sum_{i=1}^{2k+1} \frac{1}{i!} ||f^{(i)}_{\eta,2k+2}(t) \times \right.$$

$$\left. P_n((u-t)^{2k+1+i} g^{(2k+1)}(\xi), k, t)||_{L_p[x_3, Y_3]} \right\},$$

$$R_3 = \frac{1}{(2k+1)!} ||P_n(\sum_{i=1}^{2k} g^{(i)}(t)(u-t)^i(\int_t^u (u-w)^{2k+1} \times$$

$$\times (\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)dw),k,t)||_{L_p[x_3,Y_3]}$$

and

$$R_4 = \frac{1}{((2k-1)!)^2} ||P_n(g^{(2k+1)}(\xi)(u-t)^{2k+1} \times$$

$$\times (\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)dw),k,t)||_{L_p[x_3,Y_3]} \circ$$

Since $\sum_{j=0}^{k} C(j,k)d_j^{-m} = 0$, m=1,2,...,k, it follows from corollary 2.2.4 and lemma 0.6.5 , that

$$(2.3.10) \quad R_1 \leq \frac{M_8}{n^{k+1}}\{||f_{\eta,2k+2}^{(2k+1)}||_{L_p[x_3,Y_3]} + ||f_{\eta,2k+2}||_{L_p[x_3,Y_3]}\}$$

Similarly,

$$(2.3.11) \quad R_2 \leq \frac{M_9}{n^{k+1}} \{||f_{\eta,2k+2}^{(2k+1)}||_{L_p[x_3,Y_3]} + ||f_{\eta,2k+2}||_{L_p[x_3,Y_3]} \circ$$

By lemma 2.2.15 upon taking $\ell = 2k+2$,

$$(2.3.13) \quad R_3 \leq M_{10}\{n^{-(k+3/2)} ||f_{\eta,2k+2}^{(2k+2)}||_{L_p[x_2,Y_2]}$$

$$+ n^{-(2k+2)} ||f_{\eta,2k+2}^{(2k+2)}||_{L_p(I)} \}$$

and

$$(2.3.14) \quad R_4 \leq M_{11}\{n^{-(2k+2)} ||f_{\eta,2k+2}^{(2k+2)}||_{L_p[x_2,Y_2]}$$

$$+ n^{-(2k+2)} ||f_{\eta,2k+2}^{(2k+2)}||_{L_p(I)} \} .$$

Choosing n such that $n < \eta^{-2} \leq 2n$, it follows from (2.3.10-11) and lemma 0.6.5 that

$$R_1, R_2 \leq M_{12}\{n^{-1/2}\omega_{2k+1}(f, n^{-1/2}, p, [x_1, y_1] + n^{-(k+1)}\|f\|_{L_p(I)}\},$$

$$R_3, R_4 \leq M_{13}\{n^{-1/2}\omega_{2k+2}(f, n^{-1/2}, p, [x_1, y_1]) + n^{-(k+1)}\|f\|_{L_p(I)}.$$

and hence from 2.3.9,

$$(2.3.15) \quad Z_2 \leq M_{14}\{n^{-1/2}\omega_{2k+2}(f, n^{-1/2}, p, [x_1, y_1])$$

$$+ n^{-1/2}\omega_{2k+1}(f, n^{-1/2}, p, [x_1, y_1])$$

$$+ n^{-(k+1)}\|f\|_{L_p(I)}\}.$$

Also, from theorem 2.2.12 and lemma 0.6.5

$$(2.3.16) \quad Z_3 \leq \frac{M_{15}}{n^{k+1}}\|f\|_{L_p(I)}.$$

Hence, from (2.3.8), (2.3.14-15), and using again lemma 0.6.5 we get

$$(2.3.17) \quad Z_1 + Z_2 + Z_3 \leq M_{16}\{n^{-1/2}\omega_{2k+2}(f, n^{-1/2}, p, [x_1, y_1])$$

$$+ n^{-1/2}\omega_{2k+1}(f, n^{-1/2}, p, [x_1, y_1]) + n^{-(k+1)}\|f\|_{L_p(I)}\}.$$

Now, defining $\varphi^*(t) = \dfrac{\varphi(t)}{t}$, we observe that $\varphi^* \varepsilon \bar{\Phi}_r \subset \bar{\Phi}_{2k+1}$.

Also, we observe that

$$\|P_n(f, k, t) - f(t)\|_{L_p(I_1)} = O(\varphi^*(n^{-1/2})) \qquad (n \to \infty).$$

Hence, by induction hypothesis,

$$(2.3.18) \quad \omega_{2k+2}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \quad (t \to 0)$$

which, from the fact that $\varphi^* \varepsilon \ \Phi_{2k+1}$ and lemma 1.2.4, implies that

$$(2.3.19) \quad \omega_{2k+1}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \quad (t \to 0).$$

Hence, from (2.3.17-19) and the definition of φ^*, we get

$$Z_1 + Z_2 + Z_3 \leq M_{17}\{\varphi(n^{-1/2}) + n^{-(k+1)} ||f||_{L_p(I)} \}$$

which proves the theorem, since the fact $\dfrac{t^{2k+2}}{\varphi(t)} \to 0$ follows

from the fact that $\varphi \varepsilon \ \Phi_{r+1} \quad \Phi_{2k+2}$.

<u>Corollary 2.3.2</u> Let $1 \leq m \leq 2k+2$, $\varphi \varepsilon \ \bar{\Phi}_m$ and

$f \varepsilon \ L_p[0,1] \ (1 \leq p < \infty)$. Then

$$||P_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_m(f,t,p,I_2) = O(\varphi(t)) \quad (t \to 0).$$

<u>Proof</u> Follows as in the case of corollary 1.5.4.

<u>Corollary 2.3.3</u> Theorem 2.2.19

<u>Proof</u> $\varphi(t) = t^{\alpha} \varepsilon \ \Phi_{2k+2}$.

2.4 $o(\varphi)$ inverse theorems for $P_n(\cdot,k,t)$

Theorem 2.2.18 implies that, if

$$\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t)) \quad (t \to 0)$$

Then

$$|| P_n(f,k,t) - f(t) ||_{L_p(I_2)} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Correspondingly, we prove an inverse theorem for $o(\varphi)$.

<u>Theorem</u> 2.4.1 Let $\varphi \, \varepsilon \, \Phi_{2k+2}$ and $f \, \varepsilon \, L_p[0,1](1 \le p < \infty)$. Then

$$|| P_n(f,k,t) - f(t) ||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$(2.4.1) \quad \omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)) \quad (t \to 0).$$

<u>Proof</u> Choosing (x_i,y_i), $i=1,2,3,4,5$, $g \, \varepsilon \, C_o^{2k+2}$ and $fg = \bar{f}$, as in the theorem 1.5.1 and proceeding as in the theorem 2.3.1, for sufficiently small η, $\gamma > 0$, we get

$$(2.4.2) \quad || \Delta_\gamma^{2k+2} P_n(\bar{f},k,t) ||_{L_p[x_3,y_3]}$$

$$\le M_1 \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \omega_{2k+2}(\bar{f},\eta,p,[x_3,y_3]).$$

Next, we shall show that

$$(2.4.3) \quad || \Delta_\gamma^{2k+2} \{ \bar{f}(t) - P_n(\bar{f},k,t) \} ||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

After having proved (2.4.3), we define $\psi(t)$ as follows:

$$(2.4.4) \quad \psi(x) = \begin{cases} || \Delta_\gamma^{2k+2} \{ \bar{f}(t) - P_n(f,k,t) \} ||_{L_p[x_3,y_3]} \\ \qquad\qquad\qquad \text{if } x = n^{-1/2} \text{ for any } n \, \varepsilon \, \mathbb{N} \\ \psi(n^{-1/2}) \text{ if } x \, \varepsilon \, ((n+1)^{-1/2}, n^{-1/2}) \\ \qquad\qquad\qquad\qquad\qquad \text{for any } n \, \varepsilon \, \mathbb{N} \end{cases}$$

Clearly $\psi(t) = o(\varphi(t))$ and hence combining (2.4.2-4), we get

$$||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3,Y_3]} \leq M_2 \{\psi(n^{-1/2}) + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \times$$

$$\times \omega_{2k+2}(\bar{\bar{f}},\eta,p,[x_3,Y_3])\}.$$

Since this holds for all t sufficiently small and $\gamma \leq t$, choosing n such that $n \leq \eta^{-2} < n+1$, we get

$$\omega_{2k+2}(\bar{\bar{f}},t,p,[x_3,Y_3])$$

$$\cdot \leq M_3 \{\psi(\eta) + (\frac{t}{\eta})^{2k+2} \omega_{2k+2}(\bar{\bar{f}},\eta,p,[x_3,Y_3])\},$$

which, by lemma 1.2.3, implies

$$\omega_{2k+2}(\bar{\bar{f}},t,p,[x_3,Y_3]) = o(\varphi(t)) \qquad (t \to 0)$$

and since $\bar{\bar{f}} = f$ on I_2, we get

$$\omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

Hence, to complete the proof of the theorem, we are left to show that

$$(2.4.5) \quad ||P_n(\bar{\bar{f}},k,t) - \bar{\bar{f}}(t)||_{L_p[x_3,Y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty),$$

We prove this by induction as follows :

First, we prove the theorem for all $\varphi \varepsilon \Phi_1$. Later, assuming the theorem for all $\varphi \varepsilon \Phi_r$ (for some r such that $1 \leq r \leq 2k+1$), we prove the theorem for $\varphi \varepsilon \Phi_{r+1}$.

Let $\varphi \varepsilon \Phi_1$. Then, proceeding as in the proof of theorem 2.3.1 and using the fact that $t/\varphi(t) \to 0$ as $t \to 0$,

$$||P_n(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

and hence the theorem is proved, for all $\varphi \varepsilon \Phi_1$.

Now, assume that for some $r \varepsilon \mathbb{N}$ such that $1 \leq r \leq 2k+1$, the theorem holds for all $\varphi \varepsilon \Phi_r$. Let $\varphi \varepsilon \Phi_{r+1}$. Again, proceeding as in the proof of theorem 2.3.1 and choosing n such that $n \leq \eta^{-2} < n+1$, we get

$$(2.4.6) \quad ||P_n(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]} = Z_0+Z_1+Z_2+Z_3$$

where

$$(2.4.7) \quad Z_0 = ||g(t)\{P_n(f,k,t) - f(t)\}||_{L_p[x_3,y_3]}$$

$$= o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

and Z_1,Z_2,Z_3 are as before and get estimated as follows :

$$(2.4.8) \quad Z_1+Z_2+Z_3 \leq M_2\{n^{-1/2} \omega_{2k+2}(f,n^{-1/2},p,[x_1,y_1])$$

$$+ n^{-1/2} \omega_{2k+1}(f,n^{-1/2},p,[x_1,y_1]) + n^{-(k+1)}||f||_{L_p(I)}\}.$$

Now $\varphi^* = \frac{\varphi(t)}{t} \varepsilon \Phi_r \subset \Phi_{2k+1}$ and we can write

$$||P_n(f,k,t)-f(t)||_{L_p(I_1)} = o(\varphi^*(n^{-1/2})) \quad (n \to \infty).$$

Hence, by induction hypothesis

(2.4.9) $\quad \omega_{2k+2}(f,t,p,[x_1,y_1]) = o(\varphi^*(t))$ $\qquad$ $(t \to 0)$

and by lemma 1.2.5, we get

(2.4.10) $\quad \omega_{2k+1}(f,t,p,[x_1,y_1]) = o(\varphi^*(t))$ $\qquad$ $(t \to 0)$.

Again, $\varphi \in \Phi_{r+1} \subset \Phi_{2k+2}$ implies that

$$h^{2k+2} K_\varphi(h) \to 0 \quad \text{as } h \to 0$$

which implies that

(2.4.11) $\qquad t^{2k+2}/\varphi(t) \to 0 \quad \text{as } t \to 0.$

Hence, combining (2.4.6-11), we get

$$Z_0 + Z_1 + Z_2 + Z_3 = o(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

and the theorem follows.

By theorem 2.4.1 and a recursive use of lemma 1.2.5, we get the following

Corollary 2.4.2. $\quad$ Let $f \in L_p[0,1]$ $(1 \leq p < \infty)$ and $m \in \mathbb{N}$ be such that $1 \leq m \leq 2k+2$. Then, if $\varphi \in \Phi_m$,

$$||P_n(f,k,t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_m(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

2.5 $\quad$ O (φ)-inverse theorem for $P_{n,m}(.,t)$.

Theorem 2.2.18 implies that, if

$$\omega_{m+1}(f,t,p,I_1) = O(\varphi(t)) \qquad (t \to 0),$$

then

$$||P_{n,m}(f,t)-f(t)||_{L_p(I_2)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty).$$

Now, as in the case of $P_n(f,k,t)$, we prove the corresponding local $O(\varphi)$-inverse theorem for $P_{n,m}(\cdot,t)$.

Theorem 2.5.1. Let $\varphi \in \Phi_{m+1}$ and $f \in L_p(I)$ $(1 \le p < \infty)$. Then,

$$||P_{n,m}(f;t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_{m+1}(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

Proof. Choosing $g \in C_0^{m+1}$, x_i,y_i, $i = 1,2,3,4,5$ and $\bar{f}$ as in the proof of theorem 1.5.1 and proceeding as in proof of theorem 1.5.1, we get

$$||\Delta_\gamma^{m+1} P_{n,m}(\bar{f};t)||_{L_p[x_3,y_3]}$$

$$\le \gamma^{m+1} \{ ||P_{n,m}^{(m+1)}(\bar{f}-\bar{f}_{\eta,m+1};t)||_{L_p[x_3,y_3]}$$

$$+ ||P_n^{(m+1)}(\bar{f}_{\eta,m+1};t)||_{L_p[x_3,y_3]} \}.$$

Now, for sufficiently small γ, $\eta > 0$, by lemmas 2.2.14 and 0.6.5

$$||\Delta_\gamma^{m+1} P_{n,m}(\bar{f};t)||_{L_p[x_3,y_3]}$$

$$\le M_1 \gamma^{m+1} \{ n^{(m+1)/2} ||\bar{f}-\bar{f}_{\eta,m+1}||_{L_p[x_3,y_3]} + ||\bar{f}_{\eta,m+1}^{(m+1)}||_{L_p[x_3,y_3]} \}$$

$$\le \gamma^{m+1} (n^{\frac{m+1}{2}} + \frac{1}{\eta^{m+1}}) \omega_{m+1}(\bar{f},\eta,p,[x_3,y_3]).$$

Again, proceeding as for theorem 2.3.1, we are left to show that

(2.5.1) $\|\bar{f}(t)-P_{n,m}(\bar{f};t)\|_{L_p[x_3,y_3]} = O(\varphi(n^{-1/2}))$, $(n \to \infty)$,

which we first prove for $\varphi \varepsilon \Phi_1$ and then $\varphi \varepsilon \Phi_{r+1}$ assuming the result of the theorem for $\varphi \varepsilon \Phi_r$, $1 \leq r \leq m$.

Let $\varphi \varepsilon \Phi_1$. Then, for some ξ lying between u and t,

$\|P_{n,m}(fg;t) - (fg)'(t)\|_{L_p[x_3,y_3]}$

$\leq M_3 \varphi(n^{-1/2}) + \|P_{n,m}(f(u)(u-t) g'(\xi);t)\|_{L_p[x_3,y_3]}$.

A typical component of

$$Z = P_{n,m}(f(u)(u-t) g'(\xi);t)$$

can be written as

$$T_1(t) = c n^{(j-r-i)/2} \int_0^1 K(n,t,u)(t-u)^{j-r-i+1} \times$$

$$\times f(u + \frac{k}{n^{1/2}}) g'(\xi_k) du,$$

where $i,j,k,r \varepsilon \mathbb{N}'$, $0 \leq j \leq m$, $0 \leq k \leq j$, $0 \leq r \leq j-1$, $i = 0,1$, ξ_k lies between $u + k/n^{1/2}$ and t, c is a constant and $K(n,t,u)$ is given by (2.2.1),

Now, from lemma 2.2.16 and the fact that $t/\varphi(t) \to 0$ as $t \to 0$, we get that

$$\|T_1(t)\|_{L_p[x_3,y_3]} \leq M_4 \varphi(n^{-1/2}) \qquad (n \to \infty),$$

which implies that

$$||Z||_{L_p}[x_3, y_3] \leq M_5 \; \varphi(n^{-1/2}) \qquad (n \to \infty).$$

Hence the theorem is proved for all $\varphi \; \varepsilon \; \bar{\Phi}_1$.

Now, assume that the theorem is true for all $\varphi \; \varepsilon \; \bar{\Phi}_r$ $(1 \leq r \leq m'$. Let $\varphi \; \varepsilon \; \bar{\Phi}_{r+1}$. Then

$$||P_{n,m}(fg;t) - (fg)(t)||_{L_p}[x_3, y_3] \leq M_3 \; \varphi(n^{-1/2}) + Z_1 + Z_2 + Z_3$$

where

$$Z_1 = ||P_{n,m}((f(u) - f_{\eta,m+1}(u)(g(u) - g(t));t)||_{L_p}[x_3, y_3] ,$$

$$Z_2 = ||P_{n,m}((f_{\eta,m+1}(u) - f_{n,m+1}(t))(g(u) - g(t));t)||_{L_p}[x_3, y_3]$$

and

$$Z_3 = ||P_{n,m}(f_{\eta,m+1}(t)(g(u) - g(t);t)||_{L_p}[x_3, y_3] .$$

By theorem 2.2.13 and lemma 0.6.5 , we get

$$(2.5.2) \qquad Z_3 \leq M_6 \; n^{-(\frac{m+1}{2})} \; ||f||_{L_p(I)}.$$

For some ξ lying between u and t,

$$(2.5.3) \quad Z_1 = ||P_{n,m}((f(u) - f_{\eta,m+1}(u))(u-t)g'(\xi);t)||_{L_p}[x_3, y_3] .$$

A typical component of

$$P_{n,m}((f(u) - f_{\eta,m+1}(u))(u-t) \; g'(\xi);t)$$

can be written as

$$T_2(t) = c \; n^{(j-r-i)/2} \int_0^1 K(n,t,u)(t-u)^{j-r-i+1} \times$$

$$\times (f(u + \frac{k}{n^{1/2}}) - f_{\eta,m+1}(u + \frac{k}{n^{1/2}})) \; g'(\xi_k) \; du,$$

where $i,j,k,r \in \mathbb{N}^O$, $0 \leq j \leq m$, $0 \leq k \leq j$, $0 \leq r \leq j-1$, $i = 0,1$, ξ_k lies between $u + \dfrac{k}{n^{1/2}}$ and t, c is a scalar and $K(n,t,u)$ is given by (2.2.1).

Proceeding as for $T_1(t)$, we get

$$||T_2(t)||_{L_p[x_3,y_3]} \leq M_1 \{ n^{-1/2} ||f-f_{\eta,m+1}||_{L_p[x_2,y_2]}$$

$$+ n^{-\ell} ||f-f_{\eta,m+1}||_{L_p(I)} \}$$

and thus from (2.5.3),

$$Z_1 \leq M_8 \{ n^{-1/2} ||f-f_{\eta,m+1}||_{L_p[x_2,y_2]} + n^{-\ell} ||f-f_{\eta,m+1}||_{L_p(I)} \}.$$

Applying lemma 0.6.5

$$(2.5.4) \quad Z_1 \leq M_9 \{ n^{-1/2} \omega_{m+1}(f,\eta,p,[x_1,y_1]) + n^{-\ell} ||f||_{L_p(I)} \}.$$

For some ξ lying between u and t,

$$(2.5.5) \quad Z_2 \leq R_1 + R_2 + R_3 + R_4$$

where

$$R_1 = \sum_{i=1}^{m} \sum_{j=1}^{m-1} \frac{1}{i!j!} ||f_{\eta,m+1}^{(i)}(t) g^{(j)}(t) P_{n,m}((u-t)^{i+j},t)||_{L_p[x_3,y_3]},$$

$$R_2 = \frac{1}{m!} \{ \sum_{i=1}^{m} ||f_{\eta,m+1}^{(i)}(t) P_{n,m}((u-t)^{m+i} g^{(m)}(\xi),t)||_{L_p[x_3,y_3]} \},$$

$$R_3 = \frac{1}{m!} \{ \sum_{i=1}^{m-1} \frac{1}{i!} ||g^{(i)}(t) \{ P_{n,m}((u-t)^i \times$$

$$\times (\int_t^u (u-w)^m f_{\eta,m+1}^{(m+1)}(w) dw; t) \} ||_{L_p[x_3,y_3]} \}$$

and

$$R_4 = \frac{1}{(m!)^2} || P_{n,m}(g^{(m)}(\xi)(u-t)^m \int_t^u (u-w)^m f_{\eta,m+1}^{(m+1)}(w) dw; t) ||_{L_p[x_3, y_3]}.$$

After expanding Δ^j and $\prod_{i=0}^{j-1}$ a typical component of

$$P_{n,m}(g^{(m)}(\xi)(u-t)^k(\int_t^u (u-w) f_{\eta,m+1}^{(m+1)}(w) dw); t)$$

can be written as

$$(2.5.6) \quad T_3(t) = c\, n^{(\theta-k)/2} \int_0^1 K(n,t,u)(u-t)^\theta g^{(m)}(\xi_{r_2}) \times$$

$$\times \{ \int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m f_{\eta,m+1}^{(m+1)}(w) dw \} du,$$

where $T_3(t) = T_3(t,j,r_1,r_2,r_3)$, $\theta = j+r_3-r_1$, $0 \le j \le m$, $0 \le r_1 \le j-1$, $0 \le r_2 \le j$, $0 \le r_3 \le k$, ξ_{r_2} lies between $u + \frac{r_2}{n^{1/2}}$ and t, c is a constant and $K(n,t,u)$ is given by $(2.2.1)$.

Let $X(u)$ be the characteristic function of $[c,d]$ where $x_2 < c < x_3 < y_3 < d < y_2$. Then

$$(2.5.7) \quad T_3(t) = T_{31}(t) + T_{32}(t)$$

where

$$T_{31}(t) = c\, n^{\frac{(\theta-k)}{2}} \{ \int_0^1 X(u) K(n,t,u)(u-t)^\theta g^{(m)}(\xi_{r_2}) \times$$

$$\times \{ \int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m f_{\eta,m+1}^{(m+1)}(w) dw \} du \},$$

and

$$T_{32}(t) = c\, n^{(\theta-k)/2}\{\int_0^1 (1-X(u))\, K(n,t,u)\, (u-t)^\theta\, g^{(m)}\, (\xi_{r_2}\, \times$$

$$\times\ \{\int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m\, f_{\eta,m+1}^{(m+1)}(w)\, dw\}\, du\}.$$

It is clear from lemma 2.2.15 that, for sufficiently large values of n

$$||T_{31}(t)||_{L_p[x_3,Y_3]} \leq \frac{M_{10}}{n^{(k+m+1)/2}}\ ||f_{\eta,m+1}^{(m+1)}||_{L_p[x_2,Y_2]} .$$

Proceeding as in the proof of lemma 2.2.19, for any $\ell > 0$ and for all $t \in [x_3,Y_3]$,

$$|T_{32}(t)| \leq \frac{M_{11}}{n^\ell}\ ||f_{\eta,m+1}^{(m+1)}||_{L_p(I)}$$

and hence

$$||T_{32}(t)||_{L_p[x_3,Y_3]} \leq \frac{M_{12}}{n^\ell}\ ||f_{\eta,m+1}^{(m+1)}||_{L_p(I)} .$$

Thus, from (2.5.6-7), we get

$$||T_3(t)||_{L_p[x_3,Y_3]} \leq M\{n^{-(k+m+1)/2}\ ||f_{\eta,m+1}^{(m+1)}||_{L_p[x_2,Y_2]}$$

$$+\ n^{-\ell}\ ||f_{\eta,m+1}^{(m+1)}||_{L_p(I)}\},$$

which, by lemma 0.6.5, implies

$$||P_{n,m}(g^{(m)}(\xi)\,(u-t)^k(\int_t^u (u-w)^m\, f_{\eta,m+1}^{(m+1)}(w)\, dw\,;t)||_{L_p[x_3,Y_3]}$$

$$\leq M_{13}\{n^{-(k+m+1)/2}\, \eta^{-(m+1)}\, \omega_{m+1}(f,n,p,[x_1,Y_1])$$

$$+\ n^{-\ell}\, \eta^{-(m+1)}\ ||f||_{L_p(I)}\}.$$

Thus,

$$(2.5.8) \quad R_3 \le M_{14} \{ n^{-(\frac{m+2}{2})} \, \eta^{-(m+1)} \omega_{m+1}(f, \eta, p, [x_1, y_1])$$

$$+ n^{-\ell} \eta^{-(m+1)} \, ||f||_{L_p(I)} \}$$

and

$$(2.5.9) \quad R_4 \le M_{15} \{ n^{-(\frac{2m+1}{2})} \, \eta^{-(2m+1)} \omega_{m+1}(f, \eta, p, [x_1, y_1])$$

$$+ n^{-\ell} \eta^{-(m+1)} \, ||f||_{L_p(I)} \}.$$

By lemma 2.2.10 and lemma 2.2.2,

$$R_1 \le M_{16} \{ \sum_{i=1}^{m} \sum_{j=1}^{m-1} \frac{1}{n^{(i+1)/2}} \, ||f_{\eta, m+1}^{(i)}||_{L_p[x_3, y_3]} \},$$

where the summation is taken only over those i, j's which satisfy $i+j > m$. Thus, in conjunction with lemmas 0.6.3 and 0.6.5

$$(2.5.10) \quad R_1 \le \frac{M_{17}}{n^{(\frac{m+1}{2})}} \{ \frac{1}{\eta^m} \omega_m(f, \eta, p, [x_2, y_2]) + ||f||_{L_p(I)} \}.$$

A typical term in

$$P_{n,m}(g^{(m)}(\xi) \, (u-t)^{m+i}, t)$$

can be written as

$$T_4(t) = c \, n^{(j+k-r-m-i)/2} \int_0^1 K(n, t, u) \, g^{(m)}(\xi_s) \times$$

$$\times (u-t)^{j+k-r} \, du,$$

where $0 \le j \le m$, $0 \le r \le j-1$, $0 \le s \le j$, $0 \le k \le m+i$, ξ_s lies between $u + \frac{s}{n^{1/2}}$ and t, c is a constant and $K(n, t, u)$ is given by $(2.2.1)$.

Now, from corollary 2.2.5, for all $t \, \varepsilon \, [x_3, y_3]$, we obtain

$$|T_4(t)| \leq \frac{M_{18}}{n^{(m+i)/2}} \, .$$

Consequently,

$$R_2 \leq M_{19} \{ \sum_{i=1}^{m} n^{-(m+i)/2} \, ||f_{\eta,m+1}^{(i)}||_{L_p} [x_3, y_3] \} \, .$$

Again, applying lemmas 0.6.3 and 0.6.5

$$(2.5.11) \quad R_2 \leq \frac{M_{20}}{n^{(m+1)/2}} \{ \frac{1}{\eta^m} \, \omega_m(f, \eta, p, [x_2, y_2]) + ||f||_{L_p(I)} \} \, .$$

Choosing n such that $n < \eta^{-2} \leq 2n$ and taking $\ell = m+1$, from (2.5.2), (2.5.4-5) and (2.5.8-11), we get

$$(2.5.12) \quad Z_1 + Z_2 + Z_3 \leq M_{21} \{ n^{-1/2} \, \omega_{m+1}(f, n^{-1/2}, p, [x_1, y_1])$$

$$+ \, n^{-1/2} \, \omega_m(f, n^{-1/2}, p, [x_1, y_1]) + n^{-1/2} \, ||f||_{L_p(I)} \} \, .$$

Let $\varphi^*(t) = \frac{\varphi(t)}{t}$. Then $\varphi^* \, \varepsilon \, \Phi_r \subset \Phi_m$ and

$$||P_{n,m}(f,t) - f(t)||_{L_p(I_1)} = O(\varphi^*(n^{-1/2})) \quad (n \to \infty) \, .$$

Hence, by induction hypothesis,

$$(2.5.13) \quad \omega_{m+1}(f, t, p, [x_1, y_1]) = O(\varphi^*(t)) \quad (t \to 0),$$

which, by lemma 1.2.4, implies that

$$(2.5.14) \quad \omega_m(f, t, p, [x_1, y_1]) = O(\varphi^*(t)) \quad (t \to 0) \, .$$

Hence, from (2.5.12-14), we get that

$$(2.5.15) \quad Z_1 + Z_2 + Z_3 \leq M_{22} \{ n^{-1/2} \varphi^*(n^{-1/2}) + n^{-1/2} \varphi^*(n^{-1/2})$$
$$+ n^{-(\frac{m+1}{2})} \|f\|_{L_p(I)} \}.$$

Now, since $\varphi \varepsilon \ \Phi_{r+1} \subset \Phi_{m+1}$, we have

$$h^{m+1} \cdot K_\varphi(h) \to 0 \quad \text{as} \quad h \to 0$$

which implies that

$$(2.5.16) \quad t^{m+1}/\varphi(t) \to 0 \text{ as } t \to 0.$$

Hence, from (2.5.15-16), we get that

$$Z_1 + Z_2 + Z_3 \leq M_{23} \ \varphi(n^{-2/2})$$

and hence the theorem is proved.

<u>Corollary</u> 2.5.2. Let $f \ \varepsilon \ L_p[0,1]$ $(1 \leq p < \infty)$, $r \ \varepsilon \ \mathbb{N}$ be such that $1 \leq r \leq m+1$ and $\varphi \ \varepsilon \ \Phi_r$. Then

$$\|P_{n,m}(f;t) - f(t)\|_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = O(\varphi(t)) \qquad (t \to 0).$$

<u>Proof</u>. Follows along the proof of corollary 1.5.4.

Since $\varphi(t) = t^\alpha \ \varepsilon \ \Phi_{m+1}$, for any $0 < \alpha < m+1$, we get

<u>Corollary</u> 2.5.3. Theorem 2.2.20.

2.6 $o(\varphi)$-inverse theorem for $P_{n,m}(\cdot,t$

Theorem 2.2.18 implies that, if

$$\omega_{m+1}(f,t,p,I_1) = o(\varphi(t)) \qquad (t \to 0),$$

then $||P_{n,m}(f;t) - f(t)||_{L_p(I_2)} = o(\varphi(n^{-1/2}))$ $(n \to \infty)$.

The corresponding $o(\varphi)$-inverse result is

Theorem 2.6.1. Let $f \in L_p[0,1]$ $(1 \le p < \infty)$ and $\varphi \in \Phi_{m+1}$. Then

$$||P_{n,m}(f;t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$(2.6.1) \qquad \omega_{m+1}(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

Proof. Choosing (x_i,y_i), $i = 1$ to 5, $g \in C_o^{2k+2}$ and $\bar{f}$ as in theorem 1.5.1 and proceeding as in the proof of the same theorem, for sufficiently small $\eta, \gamma > 0$, we get

$$(2.6.2) \quad ||\Delta_\gamma^{m+1} P_{n,m}(\bar{f},t)||_{L_p[x_3,y_3]}$$

$$\le M_1 \gamma^{m+1}(n^{\frac{m+1}{2}} + \frac{1}{\eta^{m+1}}) \omega_{m+1}(\bar{f},\eta,p,[x_3,y_3]).$$

Next step is to prove that

$$(2.6.3) \quad ||\Delta_\gamma^{m+1} \{\bar{f}(t) - P_{n,m}(\bar{f},t)\}||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2}))(n \to \infty).$$

After having proved (2.6.3), we define $\psi(t)$ by

$$(2.6.4) \quad \psi(t) = \begin{cases} ||\Delta_\gamma^{m+1}\{\bar{f}(t) - P_{n,m}(\bar{f};t)\}||_{L_p[x_3,y_3]} & \text{if } x = n^{-1/2} \\ \psi(n^{-1/2}) & \text{if } x \in ((n+1)^{-1/2}, n^{-1/2}) \text{ for } n = 1,2,\ldots \end{cases}$$

Clearly $\psi(t) = o(\varphi(t))$ $(t \to 0)$ and

$$||\Delta_{\gamma}^{m+1} \ \bar{f}(t)||_{L_p[x_3,y_3]} \leq M_2 \{\psi(n^{-1/2})$$

$$+ \gamma^{m+1}(n^{\frac{m+1}{2}} + \frac{1}{\eta^{m+1}}) \ \omega_{m+1}(\bar{f},\eta,p,[x_3,y_3]) \,.$$

Since this holds for all sufficiently small γ , choosing n such that $n \leq \eta^{-2} < n+1$, we get

$$\omega_{m+1}(\bar{f},t,p,[x_3,y_3]) \leq M_3\{\psi(\eta)+(t/\eta)^{m+1} \ \omega_{m+1}(\bar{f},t,p,[x_3,y_3])\} \,,$$

which by lemma 1.2.5, implies that

$$\omega_{m+1}(\bar{f},t,p,[x_3,y_3]) = o(\varphi(t)) \quad (r \to 0) \,.$$

The conclusion (2.6.1) follows since $\bar{f}(t) = f(t)$ on I_2.

Now we prove (2.6.3) by an induction, as before.

Let $\varphi \ \varepsilon \ \Phi_1$. Then, for some ξ lying between u and t,

$$T = ||P_{n,m}(fg;t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq ||P_{n,m}((f(u)-f(t)) \ g(t);t)||_{L_p[x_3,y_3]}$$

$$+ ||P_{n,m}(f(u)(g(u)-g(t));t)||_{L_p[x_3,y_3]}$$

$$= ||g(t) \ \{P_{n,m}(f;t) - f(t)\}||_{L_p[x_3,y_3]}$$

$$+ ||P_{n,m}(f(u) \ (u-t)g'(\xi);t)||_{L_p[x_3,y_3]} \,.$$

Thus,

$$(2.6.5) \quad T = o(\varphi(n^{-1/2})) + ||P_{n,m}(f(u)(u-t)g'(\xi);t)||_{L_p[x_3,y_3]}$$

$$||P_{n,m}(f;t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = o(\varphi(t)) \qquad (t \to 0).$$

φ-INVERSE THEOREMS FOR LINEAR COMBINATIONS OF GENERALISED MÜLLER'S OPERATORS T_λ

3.1 Introduction.

In this chapter we study the linear combinations $T_\lambda(.,k,t)$ of the generalised Müller's operators $T_\lambda(.,t)$. These were first introduced and studied by Kunwar [32] and include several well-known operators as particular cases. The operators T_λ are defined as follows :

A non negative measurable function on $\mathbb{R}^+$ is said to be an admissible kernel function if

(1) $G(u)$ is continuous at $u = 1$,

(2)   for any $\delta > 0$, $\sup\limits_{|u-1| > \delta} G(u) < G(1)$

and (3)   there exist $\theta_1, \theta_2 > 0$ such that $(u^{-\theta_1} + u^{\theta_2}) G(u)$ is bounded on $\mathbb{R}^+$.

We denote the class of all admissible kernels by $T(\mathbb{R}^+)$. By $T_\infty(\mathbb{R}^+)$, we denote the class of all $G \in T(\mathbb{R}^+)$ which are infinite times differentiable in a neighbourhood of '1' and for which $G''(1) \neq 0$. Also, for any $\delta > 0$, T_δ denotes the class of all $G \in T_\infty(\mathbb{R}^+)$ such that supp $G \subset (1-\delta, 1+\delta)$ and G is infinite times differentiable on $(1-\delta, 1+\delta)$.

Let $G \in T(\mathbb{R}^+)$ and $\alpha \in \mathbb{R}$. Then, for $\lambda, t \in \mathbb{R}^+$, we define

$$(3.1.1) \qquad T_\lambda(f;t) = T_\lambda^{G,\alpha}(f;t) = \frac{t^{\alpha-1}}{a(\lambda)} \int_0^\infty u^{-\alpha} G^\lambda(t/u) f(u) \, du,$$

where

$$(3.1.2) \qquad a(\lambda) = \int_0^\infty u^{\alpha-2} \, G^\lambda(u) \, du,$$

whenever the above integrals exist.

Our primary interest in this chapter is to deal with $T_\lambda f$, with the kernel $G \in T_\infty(\mathbb{R}^+)$ and $f \in L_p(\mathbb{R}^+)$ $(1 \leq p < \infty)$. For $f \in L_p(\mathbb{R}^+)$, for all λ sufficiently large, $T_\lambda f \in L_p(\mathbb{R}^+)$.

The following well-known operators are particular cases of the operators T_λ.

1.   The Gamma operator G_n of Muller [43] defined by

$$G_n(f;t) = \frac{t^{n+1}}{n!} \int_0^\infty u^n \, e^{-ut} \, f(\tfrac{n}{u}) \, du,$$

is a particular case of T_λ with kernel ue^{-u}, $\alpha = 2$ and $\lambda = n$.

2.   With kernel $u^{-1} e^{-u^{-1}}$, $\lambda = n$ and $\alpha = 1$, the operators T_λ become the modified Post-Widder operators [39] defined by,

$$S_n(f;t) = \frac{1}{(n-1)!} \, (\tfrac{n}{t})^n \int_0^\infty e^{-\frac{nu}{t}} u^{n-1} \, f(u) \, du.$$

3.   Taking the kernel as in (2) above, $\alpha = 0$ and $\lambda = k$, the operator T_λ (acting on φ) gives rise to the original Post-Widder operator $L_{k,t}$ [66], defined by

$$L_{k,t}\,[f(x)] = \frac{1}{k!} \, (\tfrac{k}{t})^{k+1} \int_0^\infty e^{-\frac{nu}{t}} u^k \, \varphi(u) \, du,$$

where

$$f(x) = \int_0^\infty e^{-xu} \, \varphi(u) \, du,$$

is the Laplace transform of φ.

4.   Indeed, the above three operators can be obtained from L_λ by choosing, respectively, $p = 1$, $\alpha = 2$; $p = -1$, $\alpha = 1$ and $p = -1$, $\alpha = 0$, where L_λ is defined by

$$L_\lambda(f;t) = \frac{p\lambda^{(\lambda + \frac{\alpha-1}{p})} t^{(\lambda p + \alpha - 1)}}{\Gamma(\lambda + \frac{\alpha-1}{p})} \int_0^\infty u^{-\lambda p - \alpha} e^{-\lambda(t/u)^p} f(u)\, du,$$

where $p \in \mathbb{R}-\{0\}$, $\alpha \in \mathbb{R}$, $\lambda > 0$. The kernel, here, is $u^p e^{-u^p}$.

5.   With kernel $e^{-\frac{\log u}{\sqrt{2}\sigma}}$, $\alpha = 3/2$ and $\lambda = 1/t$, the operators T_λ reduce to the operators U_t [41] defined by

$$U_t(f;x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \int_{-\infty}^\infty f(x\, e^y) \times$$

$$\times \exp\left[-\frac{1}{2\sigma^2 t}\left(y + \frac{t\sigma}{2}\right)^2\right]\, dy, \quad (t > 0).$$

We can easily see that in all the above, the kernel belongs to $T_\infty(\mathbb{R}^+)$. It may also be remarked here that the operators T_λ share with the operators U_n of Chapter one the structural property of essentially being generated by powers of a kernel.

Kunwar [32,33] obtained an asymptotic formula of Voronovskaya type for T_λ under the assumption of the existence of $G'''(1)$ with $G''(1) \neq 0$. Under the same assumption, he determined the Lipschitz-Nikolskii constants of the operators T_λ. Later he studied the simultaneous approximation of derivatives of f by the derivative of $T_\lambda f$. Also, he obtained

local $O(\lambda^{-\alpha/2})$-inverse theorem $(0 < \alpha < 2k+2)$ and also the saturation theorem for the linear combinations $T_\lambda(.,k,t)$ (defined below) in sup-norm case.

Recently, Winslin [67] has obtained global direct, inverse and saturation theorems for the linear combinations $T_\lambda(.,k,t)$ of the operators $T_\lambda(.,t)$ in L_p-norm $(1 \leq p < \infty)$. His inverse theorems also correspond to the order $O(\lambda^{-\alpha/2})$ $(0 < \alpha < 2k+2)$.

The linear combinations $T_\lambda(.,k,t)$ are defined as follows :

Let $d_0,\ldots,d_k$ be $k+1$ positive numbers. Then the linear combinations $T_\lambda(.,k,t)$ of the operators $T_\lambda(.,t)$ are defined by

$$(3.1.3) \qquad T_\lambda(.,k,t) = \sum_{j=0}^{k} C(j,k) \; T_{d_j\lambda}(.,t),$$

where

$$(3.1.4) \qquad C(j,k) = \begin{cases} \prod_{\substack{i=0 \\ i \neq j}}^{k} \dfrac{d_j}{d_j - d_i} & , \quad k \neq 0 \\[2ex] 1 & , \quad k = 0. \end{cases}$$

In this chapter, for the operators $T_\lambda(.,k,t)$, with $G \; \varepsilon \; T_\infty(\mathbb{R}^+)$, we obtain local direct and inverse theorems for a more general order $\varphi(\lambda^{-1/2})$ in L_p-norm $(1 \leq p < \infty)$ over contracting subintervals. All through this chapter, unless stated otherwise, the operators $T_\lambda(.,t)$ corresponds to a kernel $G \; \varepsilon \; T_\infty(\mathbb{R}^+)$ and a given parameter $\alpha \; \varepsilon \; \mathbb{R}^+$.

A sectionwise summary is as follows : Section 3.2 contains basic results about the operators T_λ and $T_\lambda(.,k,t)$ which will

be used in the later sections. Section 3.3 consists of local direct theorem for $T_\lambda(.,k,t)$ and sections 3.4 and 3.5 consist of local $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operators $T_\lambda(.,k,t)$.

3.2   Basic Results.

Throughout this chapter we use the terminology that $I_j = [a_j,b_j]$, $j = 1,2$, where $0 < a_1 < a_2 < b_2 < b_1 < \infty$.

Kunwar [32] proved the following :

Lemma 3.2.1.   If $G \in T(\mathbb{R}^+)$ and $G''(1)$ exists and is non-zero then

$$\lim_{\lambda \to \infty} \frac{\lambda^{1/2} a(\lambda)}{G^\lambda(1)} = \sqrt{\pi} \left[\frac{-2G(1)}{G''(1)} \right]^{1/2} ,$$

where $a(\lambda)$ is given by $(3.1.2)$.

Immediately, we get the following

Corollary 3.2.2.   If $G \in T(\mathbb{R}^+)$ and $G''(1) \neq 0$ then

$$\frac{a(\lambda)}{a^*(\lambda)} \to 1 \quad \text{as} \quad \lambda \to \infty,$$

where $a(\lambda)$ is given by $(3.1.2)$ and

$$a^*(\lambda) = \int_0^\infty u^{\beta-2} G^\lambda(u)\, du \quad , \quad \beta \in \mathbb{R}.$$

Before stating the next theorem we define the concept of a bounding function.

Definition 3.2.3.   A continuous function $\Omega \geq 1$ defined on $\mathbb{R}^+$

is said to be   bounding with respect to the operators T_λ , if

for each compact subset K of $\mathbb{R}^+$, there exist positive numbers

λ_K and M_K such that

$$T_{\lambda_K}(\Omega;t) < M_K \quad, \; t \; \varepsilon \; K$$

and, for any bounding function Ω , we define D_Ω to be the

set of all measurable functions on $\mathbb{R}^+$ which are locally

integrable on $\mathbb{R}^+$ such that $\lim\sup\limits_{u \to 0} f(u)/\Omega(u)$, $\lim\sup\limits_{u \to \infty} f(u)/\Omega(u)$

exist.

An asymptotic expansion for the moments of the operators

T_λ is as follows :

Theorem 3.2.4. Let $G \; \varepsilon \; T(\mathbb{R}^+)$ and $k \; \varepsilon \; \mathbb{N}^0$. Then there exist

constants $C_{k,r}$, $r \geq [\frac{k+1}{2}]$ such that the following asymptotic

expansion is valid

$$t^{-k} T_\lambda ((u-t)^k, t) = \sum_{r=[\frac{k+1}{2}]}^{\infty} \frac{C_{k,r}}{\lambda^r} \quad, \; (\lambda \to \infty).$$

Lemma 3.2.5. Let $G \; \varepsilon \; T(\mathbb{R}^+)$, $G''(1)$ exist and be non-zero and

$k \; \varepsilon \; \mathbb{N}$. Then there exist λ_0 and a constant $A_k > 0$ such that

for all $\lambda > \lambda_0$ and $t \; \varepsilon \; \mathbb{R}^+$,

$$t^{-k} T_\lambda (|u-t|^k, t) \leq \frac{A_k}{\lambda^{k/2}} \; .$$

Lemma 3.2.6 : Let $G \; \varepsilon \; T(\mathbb{R}^+)$ and $m \; \varepsilon \; \mathbb{R}.$ Then

$$\lim_{\lambda \to \infty} \frac{a(\lambda-m)}{a(\lambda)} = \{G(1)\}^{-m} \; ,$$

where $a(\lambda)$ is given by (3.1.2).

The following theorem enables us to extend our results to $G \varepsilon T_\infty(\mathbb{R}^+)$ by just proving them for $G \varepsilon T_\delta$ for an appropriate $\delta > 0$.

<u>Theorem</u> 3.2.7. Let G, $G^* \varepsilon T_\infty(\mathbb{R}^+)$ such that $G = G^*$ on $(1-\delta, 1+\delta)$ for some $\delta > 0$. Then, for any fixed $\ell > 0$ and $f \varepsilon L_p(\mathbb{R}^+)$ $(1 \le p < \infty)$,

$$|| T_\lambda(f,t) - T_\lambda^*(f;t)||_{L_p(0,\infty)} = O(\lambda^{-\ell}) \quad ||f||_{L_p(\mathbb{R}^+)},$$

where T_λ, T_λ^* are the operators corresponding to the kernels G and G^*, respectively, the parameter α remaining the same.

<u>Proof.</u> Since the proof is similar to that of lemma 1.3.5, we only sketch it.

Let $\beta \varepsilon \mathbb{R}$ (we use it in the situations $\beta = \alpha$ and $\beta = \alpha+1$) and for any fixed number m, let

$$c = \int_0^\infty u^{\beta-2} G^m(u) du.$$

Indeed, if m is large enough then $c < \infty$.

Let $\sup\limits_{|u-1|> \delta} G(u) = \tau$; then $\tau < G(1)$. Hence, choose $\varepsilon > 0$ such that $\tau + \varepsilon < G(1)$. Since $G(u)$ is continuous at 1, there exists $\delta_0 > 0$ such that $\delta \ge \delta_0$ and

$$\inf\limits_{|u-1|< \delta_0} G(u) > \tau + \varepsilon.$$

Let $\gamma = \min \{(1 \pm \delta_0)^{\alpha-2} , (1 \pm \delta_0)^{\beta-2}\}.$

Now,

$$a(\lambda) = \int_0^\infty u^{\alpha-2} G^\lambda(u)\, du \geq \int_{1-\delta_0}^{1+\delta_0} u^{\alpha-2} G^\lambda(u)\, du$$

$$\geq (\tau+\varepsilon)^\lambda \, 2\delta_0\, \gamma.$$

Hence,

$$\int_{|u-1|>\delta} u^{\beta-2} G^\lambda(u)\, du = \int_{|u-1|>\delta} u^{\beta-2} G^m(u)\, G^{\lambda-m}(u)\, du$$

$$\leq \tau^{\lambda-m} \int_0^\infty u^{\beta-2} G^m(u)\, du$$

$$\leq \tau^{\lambda-m}\, c.$$

Thus,

$$\frac{1}{a(\lambda)} \int_{|u-1|>\delta} u^{\beta-2} G^\lambda(u)\, du \leq \frac{\tau^{\lambda-m}\, c}{(\tau+\varepsilon)^\lambda\, 2\delta_0\, \gamma} = O(\lambda^{-\ell}).$$

We observe that

$$\frac{\int_{|u-1|>\delta} u^{\beta-2} G^\lambda(u)\, du}{\int_{1-\delta}^{1+\delta} u^{\alpha-2} G^\lambda(u)\, du} < \frac{c}{2\,\delta_0\,\tau^m\gamma} \left(\frac{\tau}{\tau+\varepsilon}\right)^\lambda = O(\lambda^{-\ell}).$$

Hence, it is clear that

$$(3.2.2) \qquad \frac{a(\lambda)}{a^*(\lambda)} = 1 + o(\lambda^{-\ell}).$$

Now,

$$||T_\lambda(f;t) - T_\lambda^*(f;t)||_{L_p(\mathbb{R}^+)}$$

$$\leq ||\frac{1}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^\lambda(u)\, f(t/u)\, du$$

(contd.)

$$- \frac{1}{a^{*}(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^{\lambda}(u) f(t/u) du \|_{L_p(\mathbb{R}^+)}$$

$$+ \| \frac{1}{a(\lambda)} \int_{|u-1|>\delta} u^{\alpha-2} G^{\lambda}(u) f(t/u) du \|_{L_p(\mathbb{R}^+)}$$

$$+ \| \frac{1}{a^{*}(\lambda)} \int_{|u-1|>\delta} u^{\alpha-2} G^{*\lambda}(u) f(t/u) du \|_{L_p(\mathbb{R}^+)}$$

(3.2.3) $= J_1 + J_2 + J_3$, say.

From (3.2.1), it easily follows that

(3.2.4) $J_2 , J_3 = O(\lambda^{-\ell}) \|f\|_{L_p(\mathbb{R}^+)}$.

$$J_1 = \| (\frac{a(\lambda)}{a^{*}(\lambda)} - 1) \frac{1}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^{\lambda}(u) f(t/u) du \|_{L_p(\mathbb{R}^+}$$

(3.2.5) $= O(\lambda^{-\ell}) \|f\|_{L_p(\mathbb{R}^+)}$,

from (3.2.1-2) and hence combining (3.2.3-5), we get the result.

Lemma 3.2.8. Let $f \varepsilon L_p(\mathbb{R}^+)$ $(1 \le p < \infty)$ and $G \varepsilon T_\delta$ with

$0 < \delta < \min (\frac{a_2-a_1}{a_1} , \frac{b_1-b_2}{b_1})$. Then

$$\|T_\lambda(f;t)\|_{L_p(I_2)} \le (1+\delta)^{\frac{1}{p}} \|f\|_{L_p(I_1)} .$$

Proof. We can easily see from the choice of δ that if
$u \varepsilon (1-\delta, 1+\delta)$ and $t \varepsilon [a_2, b_2]$ then $t/u \varepsilon [a_1, b_1]$.

$$T_\lambda(f;t) = \frac{1}{a(\lambda)} \int_0^\infty u^{\alpha-2} G^{\lambda}(u) f(t/u) du.$$

Hence, by Jensen's inequality,

$$||T_\lambda(f;t')||^p \leq \frac{1}{a(\lambda)} \int_{a_2}^{b_2} \int_0^\infty u^{\alpha-2} G^\lambda(u) |f(t/u)|^p \, du \, dt$$

$$= \frac{1}{a(\lambda)} \int_{a_2}^{b_2} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^\lambda(u) |f(t/u)|^p \, du \, dt$$

$$\leq \frac{1}{a(\lambda)} \int_{a_1}^{b_1} \int_{1-\delta}^{1+\delta} u^{\alpha-1} G^\lambda(u) |f(t)|^p \, du \, dt$$

$$\leq ||f||_{L_p(I_1)}^p (1+\delta),$$

and hence the lemma.

Lemma 3.2.9. Let $a < c < d < b$ and $X(u)$ be the characteristic function of the interval $[a,b]$. Further, let $f \in L_p(0,\infty)$ $(1 \leq p < \infty)$ and $G \in T_\delta$ with $0 < \delta < \min(\frac{c-a}{a}, \frac{b-d}{b})$. Then, for all $t \in [c,d]$,

$$T_\lambda((1-X) f;t) = 0.$$

The following lemma gives an expansion of the derivative of the operator.

Lemma 3.2.10. Let $G \in T_\delta$ and $\lambda > m \in \mathbb{N}$. Then there holds for $u,t \in \mathbb{R}^+$,

$$\frac{\partial^m}{\partial t^m} \{t^{\alpha-1} G^\lambda(t/u)\} = t^{\alpha-1} G^{\lambda-m}(t/u) \sum_{k=0}^m \sum_{\nu=0}^{[\frac{m-k}{2}]} \lambda^{\nu+k} \times$$

$$\times \{G'(t/u)\}^k g_{\nu,k,m}(t,u),$$

where the functions $g_{\nu,k,m}(t,u)$ are certain linear combination

of products of powers of u^{-1}, t^{-1} and $G^{(k)}(t/u)$, $k = 0,1,\ldots,m$ and are independent of λ.

We close this section by stating a result about linear combinations $T_\lambda(.,k,t)$.

Theorem 3.2.11. Let $G \varepsilon T_\infty(\mathbb{R}^+)$, Ω be a bounding function for G and $f \varepsilon D_\Omega$. If at a point $t \varepsilon \mathbb{R}^+$, $f^{(2k+2)}$ exists, then there holds :

$$|T_\lambda(f;k,t) - f(t)| = O(\lambda^{-(k+1)}) ,$$

where $k = 0,1,2,\ldots$. Also, if $f^{(2k+2)}$ exists and is continuous on an open interval (c,d) containing $[a,b]$, then the result holds uniformly in $t \varepsilon [a,b]$.

3.3 Direct theorem.

In this section, we prove a direct theorem which contains both $O(\varphi)$ and $o(\varphi)$ direct theorems as corollaries.

Theorem 3.3.1. Let $f \varepsilon L_p(0,\infty)$ $(1 \leq p < \infty)$. Then

$$||T_\lambda(f,k,t) - f(t)||_{L_p(I_2)}$$

$$\leq M\{\omega_{2k+2}(f,\lambda^{-1/2},p,I_1) + \lambda^{-(k+1)} ||f||_{L_p(I)}\}.$$

Before going to the proof of the theorem, we shall prove the following results.

Lemma 3.3.2. Let $G \varepsilon T_\delta$ with $0 < \delta < \min(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_1})$ and $f \varepsilon L_p(0,\infty)$ $(1 \leq p < \infty)$ have $2k+2$ derivatives on I_1 w

$f^{(2k+1)} \varepsilon$ A.C.(I_1) and $f^{(2k+2)} \varepsilon L_p(I_1)$. Then

$$||T_\lambda(f,k,t) - f(t)||_{L_p(I_2)}$$

$$\leq \frac{M}{\lambda^{k+1}} \{||f^{(2k+2)}||_{L_p(I_1)} + ||f||_{L_p(I_1)}\}.$$

Proof. For $t \varepsilon I_2$ and $u \varepsilon I_1$, we have

$$(3.3.1) \quad f(u) = \sum_{i=1}^{2k+1} \frac{(u-t)^i}{i!} f^{(i)}(t)$$

$$+ \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w) \, dw.$$

Now, by the choice of δ, we have

$$(3.3.2) \quad T_\lambda(f;t) = \frac{1}{a(\lambda)} \int_0^\infty t^{\alpha-1} u^{-\alpha} X(u) G^\lambda(t/u) \, du,$$

where $X(u)$ is the characteristic function of I_1.

Thus, from $(3.3.1-2)$, for $t \varepsilon I_2$, we get

$$(3.3.3) \quad T_\lambda(f,k,t) - f(t) = \sum_{i=1}^{2k+1} \frac{f^{(i)}(t)}{i!} T_\lambda((u-t)^i, k, t)$$

$$+ \frac{1}{(2k+1)!} T_\lambda(\int_t^u (u-w)^{2k+1} X(u) f^{(2k+2)}(w) \, dw, k, t).$$

Since $\sum_{j=0}^k c(j,k) d_j^{-m} = 0$, $m = 1, 2, \ldots, k$, using lemmas 3.2 and 0.6.3 , we get

$$(3.3.4) \quad ||\sum_{i=1}^{2k+1} \frac{f^{(i)}(t)}{i!} T_\lambda((u-t)^i, k, t)||_{L_p(I_2)}$$

$$\leq \frac{M_1}{\lambda^{k+1}} \{||f^{(2k+2)}||_{L_p(I_2)} + ||f||_{L_p(I_2)}\}.$$

Now, to estimate

$$Z = |T_\lambda(\chi(u) \int_t^u (u-w)^{2k+1} f^{(2k+2)}(w) \, dw; t)| \, ,$$

we consider the cases $p = 1$ and $p > 1$ seperately.

Suppose $p > 1$. Then

$$Z \leq T_\lambda(\chi(u) |u-t|^{2k+2} \cdot \frac{1}{|u-t|} |\int_t^u |f^{(2k+2)}(w)| \, dw|; t)$$

$$= T_\lambda(|u-t|^{2k+2} \cdot H_f(u); t),$$

where $H_f(u)$ is the Hardy Littlewood majorant of $|f^{(2k+2)}|$ over the interval $[a_1, b_1]$. Hence, by Holder's inequality, we get

$$Z \leq \{T_\lambda(|u-t|^{(2k+2)q}; t)\}^{1/q} \{T_\lambda(\chi(u)|H_f(u)|^p; t)\}^{1/p} \, .$$

Thus, from lemmas 3.2.5 and 0.6.2 , we get

$$||Z||_{L_p(I_2)} \leq \frac{M_2}{\lambda^{k+1}} ||f^{(2k+2)}||_{L_p(I_1)} \, .$$

Hence,

$$(3.3.5) \quad ||T_\lambda(\int_t^u \chi(u)(u-w)^{2k+1} f^{(2k+2)}(w) \, dw, k, t)||_{L_p(I_2)}$$

$$\leq \frac{M_3}{\lambda^{k+1}} ||f^{(2k+2)}||_{L_p(I_1)} \, .$$

Suppose $p = 1$. Then, choose $r = r(\lambda)$ such that $r/\lambda^{1/2} \leq \max \{b_1 - b_2, b_2 - a_1\} \leq (r+1)/\lambda^{1/2}$. Then, for $t \, \varepsilon \, I_2$,

$$Z \leq \frac{1}{a(\lambda)} \int_{a_1}^{b_1} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) |u-t|^{2k+1} \times$$

$$\times |\int_{t}^{u} X(w) |f^{(2k+2)}(w)| dw| du$$

$$\leq \sum_{\ell=0}^{r} \{\frac{1}{a(\lambda)} \int_{t+\ell\lambda^{-1/2}}^{t+(\ell+1)\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) |u-t|^{2k+1} \times$$

$$\times (\int_{t}^{t+(\ell+1)\lambda^{-1/2}} X(w) |f^{(2k+2)}(w)| dw) du$$

$$+ \frac{1}{a(\lambda)} \int_{t-(\ell+1)\lambda^{-1/2}}^{t-\ell\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) |u-t|^{2k+1} \times$$

$$\times (\int_{t-(\ell+1)\lambda^{-1/2}}^{t} X(w) |f^{(2k+2)}(w) dw) du\}.$$

Let $X_{t,c,d}$ denote the characteristic function of the interval $[t-c\lambda^{-1/2}, t+d\lambda^{-1/2}]$ where $c,d \in \mathbb{N}^{o}$. Then

$$Z \leq \sum_{\ell=1}^{r} \{\frac{1}{a(\lambda)} \int_{t+\ell\lambda^{-1/2}}^{t+(\ell+1)\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \lambda^2 \ell^{-4} |u-t|^{2k+5} \times$$

$$\times (\int_{t}^{t+(\ell+1)\lambda^{-1/2}} X(w) X_{t,0,\ell+1}(w) |f^{(2k+2)}(w) dw) du$$

$$+ \frac{1}{a(\lambda)} \int_{t-(\ell+1)\lambda^{-1/2}}^{t-\ell\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^{\lambda}(t/u) \lambda^2 \ell^{-4} |u-t|^{2k+5} \times$$

$$\times (\int_{t-(\ell+1)\lambda^{-1/2}}^{t} X(w) X_{t,\ell+1,0}(w) |f^{(2k+2)}(w)| dw) du\}$$

$$\div \frac{1}{a(\lambda)} \int_{a_2 - \lambda^{-1/2}}^{b_2 + \lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^\lambda(t/u) |u-t|^{2k+1} \times$$

$$\times \int_{t-\lambda^{-1/2}}^{t+\lambda^{-1/2}} X(w) X_{\ell,1,1}(w) |f^{(2k+2)}(w)| dw \} du$$

$$\leq \sum_{\ell=1}^{r} \{ \frac{\lambda^2}{\ell^4} \cdot \frac{1}{a(\lambda)} \{ \int_{t+\ell\lambda^{-1/2}}^{t+(\ell+1)\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^\lambda(t/u) |u-t|^{2k+5} \times$$

$$\times (\int_{a_1}^{b_1} X_{t,0,\ell+1}(w) |f^{(2k+2)}(w)| dw) du$$

$$+ \int_{t-(\ell+1)\lambda^{-1/2}}^{t-\ell\lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} G^\lambda(t/u) |u-t|^{2k+5} \times$$

$$\times (\int_{a_1}^{b_1} X_{t,\ell+1,0}(w) |f^{(2k+2)}(w)| dw) du \} \}$$

$$+ \frac{1}{a(\lambda)} \int_{a_2 - \lambda^{-1/2}}^{b_2 + \lambda^{-1/2}} t^{\alpha-1} u^{-\alpha} |u-t|^{2k+1} G^\lambda(t/u) \times$$

$$\times \int_{a_1}^{b_1} X_{t,1,1}(w) |f^{(2k+2)}(w)| dw) du .$$

Now, using lemma 3.2.5, we get

$$Z \leq M_5 \lambda^{-(2k+1)/2} \{ \sum_{\ell=1}^{r} \ell^{-4} \{ \int_{a_1}^{b_2} X_{t,0,\ell+1}(w) |f^{(2k+2)}(w)| dw$$

$$+ \int_{a_1}^{b_1} X_{t,\ell+1,0}(w) |f^{(2k+2)}(w)| dw \}$$

$$+ \int_{a_1}^{b_1} X_{t,1,1}(w) |f^{(2k+2)}(w)| dw \}.$$

Thus, using Fubini's theorem,

$$||Z||_{L_p(I_2)} \leq M_5 \lambda^{-(2k+1)/2} \sum_{\ell=1}^{r} \ell^{-4} \{\int_{a_1}^{b_1} (\int_{a_1}^{b_1} X_{t,0,\ell+1}(w) \, dt) \times$$

$$\times |f^{(2k+2)}(w)| \, dw$$

$$+ \int_{a_1}^{b_1} (\int_{a_2}^{b_2} X_{t,\ell+1,0}(w) \, dt) \, |f^{(2k+2)}(w)| \, dw\}$$

$$+ \int_{a_1}^{b_1} (\int_{a_2}^{b_2} X_{t,1,1}(w) \, dt) \, |f^{(2k+2)}(w)| \, dw$$

$$= M_5 \lambda^{-(2k+1)/2} \{\sum_{\ell=1}^{r} \ell^{-4} \{\int_{a_1}^{b_1} |f^{(2k+2)}(w)| \, (\int_{w-(\ell+1)\lambda^{-1/2}}^{w} dt) \, dw$$

$$+ \int_{a_1}^{b_1} |f^{(2k+2)}(w)| \, (\int_{w}^{w+(\ell+1)\lambda^{-1/2}} dt) \, dw\}$$

$$+ \int_{a_1}^{b_1} |f^{(2k+2)}(w)| \, (\int_{w-\lambda^{1/2}}^{w+\lambda^{1/2}} dt) \, dw\}$$

$$\leq M_6 \lambda^{-(k+1)} \, ||f^{(2k+2)}||_{L_1(I_1)} ,$$

since $\sum_{\ell=1}^{\infty} \ell^{-3}$ is convergent.

Thus,

$$(3.3.6) \quad ||T_\lambda(\int_t^u (u-w)^{2k+1} X(u) \, f^{(2k+2)}(w) \, dw, k, t)||_{L_p(I_2)}$$

$$\leq M_6 \lambda^{-(k+1)} \, ||f^{(2k+2)}||_{L_1(I_1)} .$$

Hence, from (3.3.3-6), we get the lemma.

Proof of the theorem.   It is sufficient to prove the theorem for all $G \in T_\delta$ with $0 < \delta < \min(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_1})$.

We have

$$||T_\lambda(f,k,t) - f(t)||_{L_p(I_2)}$$

$$\leq ||T_\lambda(f-f_{\eta,2k+2},k,t)||_{L_p(I_2)}$$

$$+ ||T_\lambda(f_{\eta,2k+2},k,t) - f_{\eta,2k+2}(t)||_{L_p(I_2)}$$

$$+ ||f_{\eta,2k+2}(t) - f(t)||_{L_p(I_2)} .$$

Now, by lemma 3.2.9, we have

$$||T_\lambda(f-f_{\eta,2k+2},k,t)||_{L_p(I_2)}$$

$$\leq ||T_\lambda(X(f - f_{\eta,2k+2}),k,t)||_{L_p(I_2)} ,$$

where X is the characteristic function of I_1.

Thus, for sufficiently small $\eta > 0$, from lemmas 3.2.8 and 0.6.

$$(3.3.7) \quad ||T_\lambda(f-f_{\eta,2k+2},k,t)||_{L_p(I_2)} \leq M_2 \, \omega_{2k+2}(f,\eta,p,I_1)$$

and

$$(3.3.8) \quad ||f_{\eta,2k+2}(t) - f(t)||_{L_p(I_2)} \leq M_3 \, \omega_{2k+2}(f,\eta,p,I_1) .$$

Now, from lemma 3.3.2 and 0.6.5

$$(3.3.9) \quad || T_\lambda(f_{n,2k+2},k,t) - f_{n,2k+2}(t)||_{L_p(I_2)}$$

$$\leq \frac{M_3}{\lambda^{k+1}} \{n^{-(2k+2)} \omega_{2k+2}(f,n,p,I_1) + ||f||_{L_p(I_1)}\}.$$

Thus, choosing $n = \lambda^{-1/2}$ and combining (3.3.7-9) we get the theorem.

Now, $\varphi \varepsilon \Phi_{2k+2}$ implies that $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$ and hence we get the following corollaries :

<u>Corollary 3.3.3.</u>   Let $\varphi \varepsilon \Phi_{2k+2}$ and $f \varepsilon L_p(0,\infty)$ $(1 \leq p < \infty)$. Then

$$\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t)) \qquad (t \to 0)$$

implies that

$$|| T_\lambda(f,k,t) - f(t)||_{L_p(I_2)} = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$$

<u>Corollary 3.3.4.</u>   Let $\varphi \varepsilon \Phi_{2k+2}$ and $f \varepsilon L_p(0,\infty)$ $(1 \leq p < \infty)$. Then

$$\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t)) \qquad (t \to 0)$$

implies that

$$|| T_\lambda(f,k,t) - f(t)||_{L_p(I_2)} = o(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$$

3.4   O(φ)-inverse theorem.

In this section we prove the corresponding O(φ)-inverse theorem of corollary 3.3.3.

<u>Theorem 3.4.1.</u>   Let $f \varepsilon L_p(\mathbb{R}^+)$ $(1 \leq p < \infty)$ and $\varphi \varepsilon \Phi_{2k+2}$. The

$(3.4.1)$ $\quad || T_\lambda(f,k,t) - f(t) ||_{L_p(I_1)} = O(\varphi(\lambda^{-1/2}))$ $\quad (\lambda \to \infty)$

implies that

$(3.4.2)$ $\quad \omega_{2k+2}(f,t,p,I_2) = O(\varphi(t))$ $\quad (t \to 0)$.

Before proceeding to the proof of the theorem, we shall prove the following results which will be useful in the proof of the theorem.

<u>Lemma</u> 3.4.2. Let $f \in L_p(\mathbb{R}^+)$ $(1 < p < \infty)$, $i,j \in \mathbb{N}^0$ and $G \in T_\delta$ with $0 < \delta < \min(\frac{a_2 - a_1}{a_1}, \frac{b_1 - b_2}{b_1})$. Then

$$|| T_\lambda((u-t)^i \int_t^u (u-w)^j f(w) dw; t) ||_{L_p(I_2} \leq M \lambda^{-(\frac{i+j+1}{2})} ||f||_{L_p(I_1}$$

<u>Proof.</u> In view of the choice of δ, from lemma 3.2.9, we see that

$$|| T_\lambda((u-t)^i \int_t^u (u-w)^j f(w) dw; t) ||_{L_p(I_2}$$

$$= || T_\lambda((u-t)^i \chi(u) \int_t^u (u-w)^j f(w) dw; t) ||_{L_p(I_2)} ,$$

where χ is the characteristic function of I_1.

Now the proof follows exactly as in the case of the estimate of Z in the proof of lemma 3.3.2.

<u>Lemma</u> 3.4.3. Let $f \in L_p(\mathbb{R}^+)$ $(1 \leq p < \infty)$ and $G \in T_\delta$ with $0 < \delta < \min(\frac{a_2 - a_1}{a_1}, \frac{b_1 - b_2}{b_2})$. Then

$$|| T_\lambda(|u-t|^i |f(u)|; t) ||_{L_p(I_2)} \leq M \lambda^{-i/2} ||f||_{L_p(I_1)} .$$

<u>Proof.</u> Suppose $p > 1$. Then, by Holder's inequality,

$$T_\lambda(|u-t|^i \, |f(u)|;t)$$

$$\leq \{T_\lambda(|u-t|^{iq};t)\}^{1/q} \, \{T_\lambda(|f(u)|^p;t)\}^{1/q} \qquad (\tfrac{1}{p} + \tfrac{1}{q} = 1).$$

Then, from lemmas 3.2.5 and 3.2.8, we get the result.

Suppose $p = 1$. Then

$$\int_{a_2}^{b_2} |T_\lambda(|u-t|^i \, |f(u)|;t)| \, dt$$

$$\leq \int_{a_2}^{b_2} \frac{1}{a(\lambda)} \int_0^\infty u^{\alpha-2} G^\lambda(u) |f(t/u)| \, |\tfrac{t}{u} -t|^i \, du \, dt$$

$$= \frac{1}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^\lambda(u) \int_{a_2}^{b_2} |f(t/u)| \, |\tfrac{t}{u} -t|^i \, dt \, du$$

$$\leq \frac{M_1}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^\lambda(u) \, |\tfrac{1}{u} - 1|^i \int_{a_2}^{b_2} |f(t/u)| \, dt \, du$$

$$\leq \frac{M_2}{a(\lambda)} \int_{1-\delta}^{1+\delta} u^{\alpha-2} G^\lambda(u) \, |\tfrac{1}{u} - 1|^i \int_{a_1}^{b_1} |f(t)| \, dt \, du$$

and hence, from lemma 3.2.5, we get that

$$||T_\lambda(|u-t|^i \, |f(u)|;t)||_{L_1(I_2)} \leq \frac{M_2}{\lambda^{i/2}} ||f||_{L_1(I_1)}$$

and hence the result is proved.

<u>Lemma</u> 3.4.4. Let $f \in L_p(0,\infty)$ $(1 \leq p < \infty)$ and $G \in T_\delta$ with $0 < \delta < \min(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_1})$ and $\lambda > 2m \in \mathbb{N}$. Then

$$||T_\lambda^{(2m)}(f;t)||_{L_p(I_2)} = O(\lambda^m ||f||_{L_p(I_1)}).$$

If, in addition, f is 2m-times differentiable on $\mathbb{R}^+$ such that $f^{(2m-1)}$ is absolutely continuous on $\mathbb{R}^+$ and $f^{(2m)} \in L_p(I_2)$. Then

$$|| T_\lambda^{(2m)}(f;t) ||_{L_p(I_2)} \leq M || f^{(2m)} ||_{L_p(I_1)} .$$

Proof. We have

$$T_\lambda(f;t) = \frac{1}{a(\lambda)} \int_0^\infty t^{\alpha-1} u^{-\alpha} G^\lambda(t/u) f(u) \, du.$$

Hence, by lemma 3.2.10,

$$T_\lambda^{(2m)}(f;t) = \frac{1}{a(\lambda)} \int_0^\infty [t^{\alpha-1} G^\lambda(t/u)]^{(2m)} u^{-\alpha} f(u) \, du$$

$$= \frac{1}{a(\lambda)} \int_0^\infty u^{-\alpha} t^{\alpha-1} G^{\lambda-2m}(t/u) \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} \lambda^{\nu+k} (G'(t/u))^k \times$$

$$\times g_{\nu,k,m}(t,u) f(u) \, du,$$

where the functions $g_{\nu,k,m}(t,u)$ are certain linear combinations of products of u^{-1}, t^{-1} and $G^{(k)}(t/u)$, $k = 0,\dots,2m$ and are independent of λ.

Hence, using the boundedness of $g_{\nu,k,m}$ over the region of our interest,

$$\int_{a_2}^{b_2} | T_\lambda^{(2m)}(f;t) | \, dt$$

$$\leq M_1 \sum_{k=0}^{2m} \sum_{\nu=0}^{[\frac{2m-k}{2}]} \lambda^{\nu+k} \frac{1}{a(\lambda)} \int_{a_2}^{b_2} \int_0^\infty u^{-\alpha} t^{\alpha-1} G^{\lambda-2m}(t/u) \times$$

$$\times \{G'(t/u)\}^k | f(u) | \, du \, dt$$

$$= M_1 \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} \lambda^{\nu+k} \frac{1}{a(\lambda)} \int_{a_2}^{b_2} \int_0^\infty y^{\alpha-2} \ G^{\lambda-2m}(y) |G'(y)|^k |f(t/y)| \, dy \ d$$

Now using the fact that $G \ \varepsilon \ T_\delta$

$$||T_\lambda^{(2m)} \ (f;t)||_{L_1(I_2)}$$

$$= M_2 \ ||f||_{L_1(I_1)} \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} \lambda^{\nu+k} \times$$

$$\times \frac{1}{a(\lambda)} \int_0^\infty y^{\alpha-2} \ G^{\lambda-2m}(y) \ |G'(y)|^k \, dy$$

$$\leq M_2 ||f||_{L_1(I_1)} \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} \lambda^{\nu+k} \ \frac{1}{a(\lambda)} \int_0^\infty y^{\alpha-2} \ G^{\lambda-2m}(y) \times$$

$$\times |y-1|^k \ |G''(\xi)|^k \, dy,$$

for some ξ lying between y and 1, since $G'(1) = 0$. Now, usin
lemmas 3.2.5-6, we get

$$||T_\lambda^{(2m)} (f;t)||_{L_1(I_2)} \leq M_3 ||f||_{L_1(I_1)} \sum_{k=0}^{2m} \sum_{\nu=0}^{\left[\frac{2m-k}{2}\right]} \lambda^{\nu+k/2}$$

$$\leq M_4 \lambda^m \ ||f||_{L_1(I_1)}.$$

Similarly, we can prove

$$||T_\lambda^{(2m)} (f;t)||_{L_\infty(I_2)} \leq M_5 \lambda^m \ ||f||_{L_\infty(I_1)}$$

and hence we get the result from Riesz-Thorin's interpolatior
theorem.

Since $f^{(2m-1)}$ is absolutely continuous, $f^{(2m)} \varepsilon L_p(\mathbb{R}^+)$ and $G \varepsilon T_\delta$, we can write

$$T_\lambda^{(2m)}(f;t) = T_\lambda(f^{(2m)},t)$$

and hence from lemma 3.2.8, the second part of the result follows.

<u>Proof of theorem 3.4.1.</u> In view of lemma 3.2.7, it is sufficient to prove the theorem for all $G \varepsilon T_\delta$ with $0 < \delta < \min(\frac{a_2-a_1}{a_1}, \frac{b_1-b_2}{b_1})$. Choose x_i, y_i, $i = 1,2,3,4$ such that $a_1 < x_i < x_{i+1} < a_2 < b_2 < y_{i+1} < y_i < b_1$. Then

$$||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3,y_3]} \leq ||\Delta_\gamma^{2k+2}\{f(t)-T_\lambda(f,k,t)\}||_{L_p[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{2k+2} T_\lambda(f,k,t)||_{L_p[x_3,y_3]}.$$

By lemma 0.6.1

$$(3.4.3) \quad ||\Delta_\gamma^{2k+2} T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$= ||\int_0^\gamma \cdots \int_0^\gamma T_\lambda^{(2k+2)}(f,k,t + \sum_{i=1}^{2k+2} z_i)$$

$$dz_1 \cdots dz_{2k+2}||_{L_p[x_3,y_3]}.$$

Now, by repeated applications of Jensen's inequality and Fubini's theorem, for all γ sufficiently small, we have

$$\int_{x_3}^{y_3} |\int_0^\gamma \cdots \int_0^\gamma T_\lambda^{(2k+2)}(f,k,t + \sum_{i=1}^{2k+2} z_i) dz_1 \cdots dz_{2k+2}|^p dt$$

(contd.)

$$\leq \gamma^{(2k+2)(p-1)} \int_0^\gamma \cdots \int_0^\gamma ||T_\lambda^{(2k+2)}(f,k,t)||^p_{L_p[x_3,y_3]} \, dz_1 \ldots dz_{2k+2}$$

$$= \gamma^{(2k+2)p} ||T_\lambda^{(2k+2)}(f,k,t)||^p_{L_p[x_3,y_3]}.$$

Hence, from (3.4.3),

$$||\Delta_\gamma^{2k+2} T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$\leq \gamma^{(2k+2)} ||T_\lambda^{(2k+2)}(f,k,t)||_{L_p[x_3,y_3]}.$$

Suppose supp $f \subset (x_3,y_3)$. Then, from lemma 3.4.4

$$||\Delta_\gamma^{2k+2} T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$\leq \gamma^{(2k+2)} ||T_\lambda^{(2k+2)}(f-f_{\eta,2k+2},k,t)||_{L_p[x_2,y_2]}$$

$$+ ||T_\lambda^{(2k+2)}(f_{\eta,2k+2},k,t)||_{L_p[x_2,y_2]},$$

which, for all η sufficiently small, is

$$\leq M_1 \gamma^{2k+2} \{ \lambda^{k+1} ||f-f_{\eta,2k+2}||_{L_p[x_2,y_2]}$$

$$+ ||f_{\eta,2k+2}^{(2k+2)}||_{L_p[x_2,y_2]} \}.$$

Now from lemma 0.6.5 and the fact that supp $f \subset (x_3,y_3)$, we get

$$(3.4.4) \quad ||\Delta_\gamma^{2k+2} T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$\leq M_2 \gamma^{2k+2} (\lambda^{k+1} + \frac{1}{\eta^{2k+2}}) \omega_{2k+2}(f,\eta,p,[x_3,y_3]).$$

Also, since supp $f \subset (x_3, y_3)$ and γ is sufficiently small, from hypothesis (3.4.1), we get

(3.4.5) $\|\Delta_\gamma^{2k+2} \{f(t') - T_\lambda(f,k,t)\}\|_{L_p[x_3,y_3]} = O(\varphi(\lambda^{-1/2}))$ $(\lambda \to \infty)$.

Thus choosing $\lambda = \eta^{-2}$, from (3.4.4-5), we get

$\|\Delta_\gamma^{2k+2} f(t')\|_{L_p[x_3,y_3]} \leq \{\varphi(\eta) + \gamma^{2k+2}(\eta^{k+1} + \frac{1}{\eta^{2k+2}}) \times$

$$\times \ \omega_{2k+2}(f,\eta,p,[x_3,y_3])\},$$

which implies that

$$\omega_{2k+2}(f,t,p,[x_3,y_3]) \leq M_4 \{\varphi(\eta) + (\frac{t}{\eta})^{2k+2} \times$$

$$\times \ \omega_{2k+2}(f,\eta,p,[x_3,y_3])\}.$$

The conclusion, now, follows from lemma 1.2.2.

Next, we prove the theorem for general f.

Let $g \in C_0^{2k+2}$ be such that supp $g \subset (x_3,y_3)$ and $g \equiv 1$ on $[x_4,y_4]$. Then, it is sufficient to prove that

(3.4.6) $\|T_\lambda(fg,k,t') - (fg)(t)\|_{L_p[x_3,y_3]} = O(\varphi(\lambda^{-1/2}))$ $(\lambda \to \infty)$.

We prove this by induction as follows : First we prove the result for all $\varphi \in \Phi_1$. Later, assuming the theorem for all $\varphi \in \Phi_r$, for some r such that $1 \leq r \leq 2k+1$, we prove it for all $\varphi \in \Phi_{r+1}$.

Let $\varphi \in \Phi_1$. Then

$(3.4.7)$ $||T_\lambda(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]}$

$$\leq ||T_\lambda((f(u)-f(t)) \cdot g(t),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||T_\lambda(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]} .$$

Thus, using the hypothesis $(3.4.1)$ and lemma 3.4.3 for the first and second terms on the right hand side of $(3.4.7)$, we get $(3.4.6)$, since $t = o(\varphi(t))$ $(t \to 0)$. Hence the theorem is proved for all $\varphi \varepsilon \Phi_1$.

Assume that for some r such that $1 \leq r \leq 2k+1$, the result is true for all $\varphi \varepsilon \Phi_r$. Let $\varphi \varepsilon \Phi_{r+1}$. Then,

$$\varphi^*(t) = \frac{\varphi(t)}{t} \varepsilon \Phi_r .$$

Hence, by induction hypothesis, we get

$$\omega_{2k+2}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0)$$

which, by lemma 1.2.4, implies that

$$\omega_{2k+1}(f,t,p,[x_1,y_1]) = O(\varphi^*(t)) \qquad (t \to 0).$$

Hence, for j = 2k+1, 2k+2

$(3.4.8)$ $\lambda^{-\frac{1}{2}} \omega_j(f,\lambda^{-1/2},p,[x_1,y_1]) = O(\varphi(\lambda^{-1/2}))$ $(\lambda \to \infty)$.

Now, in view of $(3.4.7)$, it is sufficient to prove that

$(3.4.9)$ $J = ||T_\lambda(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]} = O(\varphi(\lambda^{-1/2}))$

$$(\lambda \to \infty).$$

We have

$(3.4.10)$ $J \leq J_1 + J_2 + J_3$, where

$$J_1 = ||T_\lambda((f(u)-f_{\eta,2k+2}(u)) (g(u) - g(t)),k,t)||_{L_p[x_3,y_3]},$$

$$J_2 = ||T_\lambda((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t)) (g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

and

$$J_3 = ||T_\lambda(f_{\eta,2k+2}(t) (g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}.$$

For some ξ lying between u and t,

$$J_1 = ||T_\lambda((f(u)-f_{\eta,2k+2}(u)) (u-t) g'(\xi),k,t)||_{L_p[x_3,y_3]}.$$

Now, choosing $\lambda = \eta^{-2}$, from lemmas 3.4.3 and 0.6.5 , we see that

$(3.4.11)$ $J_1 \leq \lambda^{-1/2} \omega_{2k+2}(f,\lambda^{-1/2},p,[x_1,y_1])$,

which, in view of $(3.4.8)$, implies that

$(3.4.12)$ $J_1 = O(\varphi(\lambda^{-1/2}))$ $(\lambda \to \infty)$.

Also by theorem 3.2.11 and lemma 0.6.5

$(3.4.13)$ $J_3 \leq \dfrac{M_5}{\lambda^{k+1}}$.

Since $\varphi \varepsilon \Phi_{r+1} \subset \Phi_{2k+2}$, $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$. Hence,

$(3.4.14)$ $J_3 = O(\varphi(\lambda^{-1/2}))$ $(\lambda \to \infty)$.

Again for some ξ lying between u and t,

$$(f_{\eta,2k+2}(u) - f_{\eta,2k+2}(t))\ (g(u) - g(t))$$

$$= \{\sum_{i=1}^{2k+1} \frac{(u-t)^i}{i!} f_{\eta,2k+2}^{(i)}(t) + \frac{1}{(2k+1)!} \int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\,dw\} \times$$

$$\times \{\sum_{i=1}^{2k} \frac{(u-t)^i}{i!}\ g^{(i)}(t) + \frac{(u-t)^{2k+1}}{(2k+1)!} g^{(2k+1)}(\xi)\}.$$

Therefore,

$$J_2 \le \frac{1}{(2k+1)!} \sum_{i=1}^{2k} ||\frac{g^{(i)}(t)}{i!}\ T_\lambda((u-t)^i \times$$

$$\times \int_t^u (u-w)^{2k+1}\ f_{\eta,2k+2}^{(2k+2)}(w)\ dw,k,t)||_{L_p[x_3,y_3]}$$

$$+ \frac{1}{((2k+1)!)^2} ||T_\lambda(g^{(2k+1)}(\xi)(u-t)^{2k+1} \times$$

$$\times \int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\,dw,k,t)||_{L_p[x_3,y_3]}$$

$$+ \{\sum_{i=1}^{2k+1} \sum_{j=1}^{2k} \frac{1}{i!j!}\ ||f_{\eta,2k+2}^{(i)}(t)\ g^{(j)}(t) T_\lambda((u-t)^{i+j},k,t)||_{L_p[x_3,y_3}$$

$$+ \frac{1}{(2k+1)!}\{\sum_{j=1}^{2k+1} \frac{1}{i!}||f_{\eta,2k+2}^{(i)}(t) \times$$

$$\times T_\lambda((u-t)^{2k+1+i}\ g^{(2k+1)}(\xi),k,t)||_{L_p[x_3,y_3]}$$

$$(3.4.15) \quad = R_1 + R_2 + R_3 + R_4,\ \text{say.}$$

By lemma 3.4.2,

$$R_1 \le M_6 \{\sum_{i=1}^{2k} \lambda^{-(k+1+i/2)}\ ||f_{\eta,2k+2}^{(2k+2)}||_{L_p[x_3,y_3]}\}$$

and

$$R_2 \leq M_7 \{\lambda^{-(2k+3/2)} \|f^{(2k+2)}_{\eta,2k+2}\|_{L_p}[x_2,y_2]\}.$$

By lemma 0.6.5 and choosing $\lambda = \eta^{-2}$ we get

$$(3.4.16) \quad R_1, R_2 \leq M_8 \lambda^{-1/2} \omega_{2k+2}(f,\lambda^{-1/2},p,[x_1,y_1])$$

which, by (3.4.8) implies that

$$(3.4.17) \quad R_1, R_2 = O(\varphi(\lambda^{-1/2})) \quad (\lambda \to \infty).$$

It follows from theorem 3.2.4 and the fact that $\sum_{j=0}^{k} C(j,k) d_j^{-m} = 0$, $m = 1,2,\ldots,k$, that

$$R \leq \frac{M_9}{\lambda^{k+1}} (\sum_{i=1}^{2k+1} \|f^{(i)}_{\eta,2k+2}\|_{L_p}[x_3,y_3]).$$

Clearly,

$$R_4 \leq \frac{M_{10}}{\lambda^{k+1}} (\sum_{i=1}^{2k+1} \|f^{(i)}_{\eta,2k+2}\|_{L_p}[x_3,y_3]).$$

Now using lemma 0.6.3 , we get

$$R_3, R_4 \leq \frac{M_{11}}{\lambda^{k+1}} \{\|f^{(2k+1)}_{\eta,2k+2}\|_{L_p}[x_3,y_3] + \|f_{\eta,2k+2}\|_{L_p}[x_3,y_3]\}$$

which, by lemma 0.6.5 , after taking $\lambda = \eta^{-2}$, implies that

$$(3.4.18) \quad R_3, R_4 \leq \lambda^{-1/2} \omega_{2k+1}(f,\lambda^{-1/2},p,[x_1,y_1])\lambda^{-(k+1)}\|f\|_{L_p}($$

Now, from (3.4.8) and the fact that $\dfrac{\lambda^{-(k+1)}}{\varphi(\lambda^{-1/2})} \to 0$ as $\lambda \to \infty$

we get that

$$(3.4.19) \quad R_3, R_4 = O(\varphi(\lambda^{-1/2}))$$

Hence, from (3.4.15), (3.4.17) and (3.4.19),

$$J_2 = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$$

Thus, $J = J_1 + J_2 + J_3 = O(\varphi(\lambda^{-1/2}))$ $\qquad (\lambda \to \infty)$

and hence the result follows.

<u>Corollary</u> 3.4.5. Let $f \varepsilon L_p(\mathbb{R}^+)$ and for some $r \varepsilon \mathbb{N}$ such that $1 \le r \le 2k+2$, $\varphi \varepsilon \Phi_r$. Then

$$||T_\lambda(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = O(\varphi(\lambda^{-1/2})).$$

<u>Proof.</u> Recursive use of lemma 1.2.4 with theorem 3.4.1.

3.5 $o(\varphi)$-inverse theorem.

In this section we prove the inverse theorem corresponding to corollary 3.3.4.

<u>Theorem</u> 3.5.1. Let $f \varepsilon L_p(\mathbb{R}^+)$ $(1 \le p < \infty)$ and $\varphi \varepsilon \Phi_{2k+2}$. Then

(3.5.1) $||T_\lambda(f,k,t) - f(t)||_{L_p(I_1)} = o(\varphi(\lambda^{-1/2}))$ $\qquad (\lambda \to \infty)$

implies that

(3.5.2) $\omega_{2k+2}(f,t,p,I_2) = o(\varphi(t))$ $\qquad (t \to 0).$

<u>Proof.</u> Choose x_i, y_i, $i = 1,2,3,4$ such that $a_1 < x_i < x_{i+1} < a_2 < b_2 < y_{i+1} < y_i < b_1.$

By hypothesis (3.5.1), if γ is sufficiently small and supp $f \subset (x_3, y_3)$ then

$$(3.5.3) \quad ||\Delta_\gamma^{2k+2}\{f(t) - T_\lambda(f,k,t)\}||_{L_p[x_3,y_3]} = o(\varphi(\lambda^{-1/2})) \quad (\lambda \to \infty).$$

Now, defining

$$\psi(x) = ||\Delta_\gamma^{2k+2}\{f(t) - T_\lambda(f,k,t)\}||_{L_p[x_3,y_3]},$$

where $x = \lambda^{-1/2}$, we observe that

$$\psi(t) = o(\varphi(t)) \quad (t \to 0).$$

Also, from (3.4.4) of theorem 3.4.1, we have

$$(3.5.4) \quad ||\Delta_\gamma^{2k+2} T_\lambda(f,k,t)||_{L_p[x_3,y_3]}$$

$$\leq M_1 \ \gamma^{2k+2} (\lambda^{k+1} + \frac{1}{\eta^{2k+2}}) \ \omega_{2k+2}(f,\eta,p,[x_3,y_3]).$$

Hence, combining (3.5.3-4) and choosing $\lambda = \eta^{-2}$, we get

$$||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3,y_3]} = M_2 \{\psi(\eta) + \gamma^{2k+2} (\lambda^{k+1} + \frac{1}{\eta^{2k+2}}) \times$$

$$\times \ \omega_{2k+2}(f,t,p,[x_3,y_3])\},$$

which, by lemma 1.2.3, after taking supremum over all $\gamma \leq t$, implies the result.

Thus we have proved the theorem for all f such that supp $f \subset (x_3, y_3)$.

Hence, to prove the theorem for general f, it is sufficient to prove that

(3.5.5) $||(fg)(t) - T_\lambda(fg,k,t)||_{L_p[x_3,Y_3]} = o(\varphi(\lambda^{-1/2}))$ $(\lambda \to 0)$,

where $g \in C_o^{2k+2}$ such that supp $g \subset (x_3,Y_3)$ and $g \equiv 1$ on $[x_4,Y_4]$.
Now, if $\varphi \in \Phi_1$, we have

$||T_\lambda(fg,k,t) - (fg)(t)||_{L_p[x_3,Y_3]}$

$\leq ||T_\lambda((f(u)-f(t))\ g(t),k,t)||_{L_p[x_3,Y_3]}$

$+ ||T_\lambda(f(u)(g(u)-g(t)),k,t)||_{L_p[x_3,Y_3]}$,

$= o(\varphi(\lambda^{-1/2}))$,

by (3.5.1) and lemma 3.4.3. Hence the theorem is proved for
all $\varphi \in \Phi_1$.

Next, assume the result for all $\varphi \in \Phi_r$, for some r such
that $1 \leq r \leq 2k+1$, and let $\varphi \in \Phi_{r+1}$. Let (3.5.1) hold.

Then, since $\varphi^*(t) = \dfrac{\varphi(t)}{t} \in \Phi_r$, by our assumption, we get

(3.5.6) $\omega_{2k+2}(f,t,p,[x_1,Y_1]) = o(\varphi^*(t))$ $(t \to 0)$

which, by lemma 1.2.5, implies that

(3.5.7) $\omega_{2k+1}(f,t,p,[x_1,Y_1]) = o(\varphi^*(t))$ $(t \to \infty)$.

Clearly, by hypothesis (3.5.1), we have

$||T_\lambda((f(u)-f(t))g(t),k,t)||_{L_p[x_3,Y_3]} = o(\varphi(\lambda^{-1/2}))$ $(\lambda \to \infty)$.

Hence,

$$J = ||T_\lambda(fg,k,t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq ||T_\lambda((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||T_\lambda((f(u)-f_{\eta,2k+2}(u))(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||T_\lambda((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t))(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$+ ||T_\lambda(f_{\eta,2k+2}(t)(g(u)-g(t)),k,t)||_{L_p[x_3,y_3]}$$

$$(3.5.8) \qquad = o(\varphi(\lambda^{-1/2})) + J_1 + J_2 + J_3, \text{ say.}$$

For some ξ lying between u and t,

$$J_1 \leq ||T_\lambda((f(u)-f_{\eta,2k+2}(u))(u-t)g'(\xi),k,t)||_{L_p[x_3,y_3]},$$

which after choosing $\lambda = \eta^{-2}$, from lemmas 3.4.3 and 0.6.5, implies that

$$J_1 \leq \lambda^{-1/2}\omega_{2k+1}(f,\lambda^{-1/2},p,[x_1,y_1]).$$

Applying, theorem 3.2.11 and lemma 0.6.5, we get

$$J_3 \leq \frac{M_3}{\lambda^{k+1}}.$$

Hence, using (3.5.6) and the fact that $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$ respectively, for J_1, J_3, we get

$$(3.5.9) \qquad J_1, J_3 = o(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty).$$

The quantity J_2 has already been considered and estim in (3.4.15-17) of the proof of theorem 3.4.1 as

$$J_2 \le R_1 + R_2 + R_3 + R_4 \text{ , where}$$

$$R_1, R_2 \le \lambda^{-1/2} \omega_{2k+2}(f, \lambda^{-1/2}, p, [x_1, y_1])$$

and

$$R_3, R_4 \le \lambda^{-1/2} \omega_{2k+1}(f, \lambda^{-1/2}, p, [x_1, y_1]) + \lambda^{-(k+1)} ||f||_{L_p(I_1)} .$$

Hence, from (3.5.6-7) and the fact that $t^{2k+2}/\varphi(t) \to 0$ as $t \to 0$, we get, for $i = 1, 2, 3, 4$

$$R_i = o(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty) .$$

Thus $J_2 = o(\varphi(\lambda^{-1/2}))$ $(\lambda \to \infty)$ and hence the result follows from (3.5.8-9).

An immediate consequence of this theorem and lemma 1.2.5 is the following

Corollary 3.5.2. Let $f \in L_p(\mathbb{R}^+)$ and for some $r \in \mathbb{N}$, such that $1 \le r \le 2k+2$, $\varphi \in \Phi_r$. Then

$$||T_\lambda(f, k, t) - f(t)||_{L_p(I_1)} = o(\varphi(\lambda^{-1/2})) \qquad (\lambda \to \infty)$$

implies that

$$\omega_r(f, t, p, I_2) = o(\varphi(t)) \qquad (t \to 0) .$$

$\varphi -$ INVERSE THEOREMS FOR LINEAR COMBINATIONS AND INTERPOLATORY
MODIFICATIONS OF REGULAR EXPONENTIAL TYPE OPERATORS

4.1 Introduction.

In this chapter, we study the linear combinations
$S_n(.,k,t)$ and interpolatory modifications $S_{n,m}(.,t)$ of
the regular exponential type operators $S_n(.,t)$. These
regular exponential type operators were, first, introduced
by May [39] , for approximation of functions belonging
to the class $C(A,B)$ of bounded, continuous functions on
(A,B) $(-\infty \leq A < B \leq \infty)$ into $C^{\infty}(A,B)$, having the form

$$S_n(f,t) = \int_A^B W(n,t,u)\ f(u)du,$$

where $S_n(1,t) = 1$, $t \varepsilon (A,B)$, $f \varepsilon C(A,B)$, $W(n,t,u) \geq 0$
and satisfies the following:

1. $\frac{\partial}{\partial t} W(n,t,u) = \frac{n}{p(t)} W(n,t,u)(u-t), u,t \varepsilon (A,B),$

where $p(t)$ is a polynomial of degree ≤ 2 and $p(t) > 0$
for $t \varepsilon (A,B)$.

2. For $r \varepsilon \mathbb{N}$,

$$\frac{d^r}{dt^r} S_n(f;t) = \int_A^B \{\frac{\partial^r}{\partial t^r} W(n,t,u)\}\ f(u)du .$$

The integrals in above are in the Lebesgue-Stieltjes
sense for which $W(n,t,u)$ is the distributional kernel.

3.   The kernel $W(n,t,u)$ (regarded as a function in t and u)
is measurable on $(A,B) \times (A,B)$ and

$$\int_A^B W(n,t,u) dt = a(n) , \quad u \in (A,B)$$

where $a(n)$ is a rational function of n with $a(n) \to 1$
as $n \to \infty$ and for each fixed $u \in (A,B)$, $m \in IN^o$ and n
sufficiently large

$$t^m p(t) W(n,t,u) \to 0 \quad as \quad t \to A \text{ or } B.$$

The class of operators which satisfy (1) and (2) above
are called exponential type operators.   In addition, if
they satisfy (3) also, they are called regular.   It has
recently been shown by Sinha [61] that for $f \in L_p[A,B]$
$(1 \leq p < \infty)$, regular exponential type operators constitute
an approximation method.   Well known examples of regular
exponential type operators are the operators of Post-Widder
and Gauss-Weierstrass, defined respectively by

$$S_n(f;t) = \frac{1}{\Gamma(n)} \left(\frac{n}{t}\right)^n \int_0^\infty e^{-nu/t} u^{n-1} f(u) du$$

and

$$S_n(f;t) = \left(\frac{n}{2\pi}\right)^{1/2} \int_{-\infty}^\infty e^{-n(u-t)^2/2} f(u) du.$$

Examples of exponential type operators which are not regular
are the Bernstein polynomials, the Szász operators and the
Baskakov operators, defined respectively by

$$S_n(f; t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f(\frac{k}{n}) ,$$

$$S_n(f; t) = e^{-nt} \sum_{k=0}^{\infty} \frac{1}{k!} (nt)^k f(\frac{k}{n})$$

and

$$S_n(f; t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \theta_n^{(k)}(t) t^k f(\frac{k}{n}) ,$$

where the family $\{\theta_n\}$ satisfies certain appropriate conditions.

Also, the operators $L_n(f; t)$ of Rathore [48] defined by

$$L_n(f; t) = (1+t)^{-(n+1)} \sum_{s=0}^{\infty} \binom{n+s}{s} (\frac{t}{1+t})^s f(\frac{s}{n+p})$$

give rise to the operators

$$S_n(f; t) = (1+t)^{-n} \sum_{s=0}^{\infty} \binom{n+s-1}{s} (\frac{t}{1+t})^s f(\frac{s}{n}) ,$$

where $S_n \equiv L_{n-1}$ with $p = 1$, which are of exponential type but are not regular.

All through this chapter it will be assumed that $\{S_n\}$ are regular exponential type operators and that $f \in L_p[A, B]$.

Let $d_0, d_1, \ldots, d_k$ be $k+1$ distinct positive integers. Then the linear combination $S_n(f; k, t)$ is defined by

$$S_n(f, k, t) = \sum_{j=0}^{k} C(j, k) S_{d_j n}(f, t) ,$$

where

$$C(j,k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i} , \quad k \neq 0.$$

and

$$C(0,0) = 1 .$$

The interpolatory modification $S_{n,m}(\cdot,t)$, introduced by Sinha [61] , of the regular exponential type operators $S_n(\cdot,t)$, is a modification of $S_n(\cdot,t)$, slightly different from the case of $P_{n,m}(\cdot,t)$, obtained by replacing function value $f(u)$ at the point 'u' by Newton's interpolation polynomial of $(m+1)$th degree based at the nodes $u, u + n^{-1/2}, \ldots, u + mn^{-1/2}$ and devided by $(|u-t|^{m_o+2} + 1)$, $m+1 \leq m_o$. The latter operators, thus, become L_p-bounded (as discussed by Sinha [61]).

Thus, for $f \in L_p[A,B]$, interpolatory modification, $S_{n,m}(f;t)$, of order m, of regular exponential type operators $S_n(f;t)$ is defined as follows:

$$S_{n,m}(f;t) = \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} \times$$

$$\times \left\{ \sum_{j=0}^{m} \frac{n^{j/2}}{j!} \left(\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}}) \right) \Delta^j f(u) \right\} du,$$

where $m \leq m_o$ and Δ denotes $\Delta_{n^{-1/2}}$.

The class of exponential operators was introduced by May [39] . Giving a unified treatment for this class, May [39] obtained sup-norm local inverse theorems for Butzer type linear combinations [14] of members of this class. With the additional assumption of regularity he also obtained saturation theorems for these linear combinations This treatment is based on intermediate space approach involving a Peetre's K-functional for inverse theorems and is distribution theory oriented for saturation theorems.

Later, Agrawal [1] studied the inverse and saturation theorems in simultaneous approximation for exponential type operators. Infact , he considered the problem in a more general frame work of two types of iterative combinations which contain the iterative combinations of Micchelli as well as the ordinary linear combinations considered by May [39] as special cases.

As noted before, Sinha [61] proved that regular exponential type operators are L_p-approximation methods. He, then, obtained local direct, inverse and saturation theorems with respect to L_p-norm ($1 \leq p < \infty$ for the linear combinations and interpolatory modifications of regular exponential type operators. He proved inverse theorems for the order $O(n^{-\alpha/2})$ ($0 < \alpha < 2k+2$).

In this chapter, we extend the results of Sinha [61] to a more general order $\varphi(n^{-1/2})$, which generalise the

inverse theorems for linear combinations as well as inter-
polatory modifications of regular exponential type operators
in L_p-norm $(1 \leq p < \infty)$. These are also local in nature over
contracting sub-intervals.

The layout of this chapter is as follows:
Section 4.2 contains some basic results which shall be used
in later sections. In section 4.3, we prove $O(\varphi)$-inverse
theorem for the operators $S_n(.,k,t)$ and the $o(\varphi)$-inverse
theorem for $S_n(.,k,t)$ is proved in section 4.4. Similarly,
$O(\varphi)$ and $o(\varphi)$-inverse theorems for the operators $S_{n,m}(.,t)$
are proved, respectively, in sections 4.5 and 4.6.

4.2   Basic Results.

This section contains results which are useful in the
later sections of this chapter. Throughout the rest of this
chapter, we denote $I = [A,B]$, $I_j = [a_j, b_j]$, $j = 1,2,3,$
where $A < a_j < a_{j+1} < b_{j+1} < b_j < B$.

The following theorem of Sinha [61] , establishes the
convergence of the operators $S_n(.,t)$.

Theorem 4.2.1.   Let $f \varepsilon L_p(I)$ $(1 \leq p < \infty)$. Then

$$|| S_n(f,t) - f(t) ||_{L_p[A,B]} = o(1) \quad (n \to \infty).$$

A formula for the moments of exponential type
operators [39, 1] is given by the following

Lemma 4.2.2. For $m \varepsilon \mathbb{N}$,

$$A_m(n,t) = n^m \int_A^B W(n,t,u) (u-t)^m du$$

is a polynomial in t of degree $\leq m$, in n of degree $[\frac{m}{2}]$. The coefficient of n^m in $A_{2m}(n,t)$ is $(2m-1)!! \, p^m(t)$ and the coefficient of n^m in $A_{2m+1}(n,t)$ is $C_m p^m(t)p'(t)$ where $C_m = \frac{(2m+1)!!}{3} m$.

Corollary 4.2.3. Let r be a positive real number. Then, for all t belonging to a compact subset K of (A,B) there holds

$$|A_r(n,t)| \leq M \, n^{r/2} \quad ;$$

M is a constant independent of n.

   Sinha [61] obtained the following lemma regarding moment expansion of the operators $S_{n,m}(\cdot,t)$.

Lemma 4.2.4.   Let K be a compact subset of (A,B). Then, for $k \varepsilon \mathbb{N}$ there hold :

1.   If $k \leq m$, $S_{n,m}((u-t)^k,t) = 0$.

2. $S_{n,m}((u-t)^{m+1},t) = (-1)^m \, S_n(\prod_{i=0}^{m} (u-t+\frac{i}{n^{1/2}}); t)$

$$+ 0(\frac{1}{n^{m+\frac{3}{2}}}) \quad (n \to \infty),$$

uniformly in $t \varepsilon K$.

3.   If $k > m+1$,

$$S_{n,m}((u-t)^k;t) = \sum_{r=0}^{k-1} \frac{a_r}{n^{r/2}} S_n((u-t)^{k-r};t)$$

$$+ O(n^{-(m+k+2)/2}) \quad (n \to \infty),$$

uniformly in $t \in K$ where a_r's are certain constants.

The following is an immediate corollary of the above lemma

Corollary 4.2.5.

$$S_{n,m}((u-t)^{m+1};t) = (-1)^m \frac{Q_{m+1}(t)}{n^{(m+1)/2}} + o(\frac{1}{n^{(m+1)/2}})$$

as $n \to \infty$, where $Q_{m+1}(t)$ is a polynomial in t of degree $\leq m+1$ and $Q_{m+1}(t) > 0$ in (A,B). The o-term holds uniformly with respect to $t \in K$.

Definition 4.2.6. A positive function $\psi \in C(A,B)$ is said to be a growth test function for $\{S_n\}$ if for any compact subset K of (A,B) there exists an $n_o \in \mathbb{N}$ and a positive constant M such that

$$S_n(\psi^2;t) < M, \quad n > n_o, \quad t \in K.$$

Lemma 4.2.7. Let $f \in L_p[A,B](1 \leq p < \infty)$. Then, for some constant M

$$||S_{n,m}(f;t)||_{L_p[A,B]} \leq M||f||_{L_p[A,B]}.$$

Regarding the convergence of the operator $S_{n,m}(\cdot,t)$, Sinha [61] proved the following

Theorem 4.2.8. Let $f \in L_p[A,B](1 \leq p < \infty)$. Then

$$||S_{n,m}(f,t) - f(t)||_{L_p[A,B]} = o(1) \quad (n \to \infty).$$

May [39] proved the following asymptotic formula for linear combinations of exponential type operators.

<u>Lemma 4.2.9.</u>   Let $|f(t)| \leq \psi(t)$, $t \in (A,B)$ for some growth test function ψ. If for some $t \in (A,B)$, $f^{(2k+2)}(t)$ exists then

$$(4.2.1) \quad n^{k+1}\{S_n(f,k,t)-f(t)\} = \sum_{j=k+1}^{2k+2} Q(j,k,t)f^{(j)}(t) + o(1),$$

where $Q(j,k,t)$'s are certain polynomials in t. Moreover, $Q(2k+2,k,t) = c_1(p(t))^{k+1}$ and $Q(2k+1,k,t) = c_2(p(t))^k p'(t)$, c_1, c_2 being constants. Further, if $f \in C^{2k+2}[A,B]$, then (4.2.1) is uniform in every interval $[a,b] \subset (A,B)$.

Sinha [61] proved a similar result about the operators $S_{n,m}(\cdot,t)$.

<u>Theorem 4.2.10.</u>   Let $f \in C_0^{m+1}$ with supp $f \subset I_1$. Then

$$S_{n,m}(f;t)-f(t) = \frac{(-1)^m Q_{m+1}(t)}{(m+1)! \, n^{(m+1)/2}} f^{(m+1)}(t)$$
$$+ o(\frac{1}{n^{(m+1)/2}}) \qquad (n \to \infty)$$

and

$$S_{n,m+1}(f;t) - f(t) = o(\frac{1}{n^{(m+1)/2}}) \qquad (n \to \infty),$$

uniformly in $t \in I_1$, where $Q_{m+1}(t)$ is a polynomial in t of degree $\leq m+1$ and $Q_{m+1}(t) > 0$ on (A,B).

We state here a technical lemma of Sinha [61], which is of use in our inverse theorem.

Lemma 4.2.11. Let $h \varepsilon L_p[A,B]$ $(1 \leq p < \infty)$. Then for any fixed positive number ℓ,

1. $||S_{n,m}(|u-t| \, |h(u)| \; ; \; t)||_{L_p(I_2)}$

$$\leq M_1 \{ n^{-1/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p[A,B]} \}$$

and if, in addition, supp $h \subset (A,B)$, then for $i \varepsilon \mathbb{N}^0$

2. $||S_n(|u-t|^i \, |h(u)| \; ; t)||_{L_p(I_2)}$

$$\leq M_2 \{ n^{-i/2}||h||_{L_p(I_1)} + n^{-\ell}||h||_{L_p[A,B]} \},$$

where M_1, M_2 are constants independent of n and h.

Sinha [61] obtained Bernstein-type inequalities for the operators $S_n(\cdot, t)$ and $S_{n,m}(\cdot, t)$.

Lemma 4.2.12. Let $f \varepsilon L_p[A,B]$ $(1 \leq p < \infty)$ with supp $f \subset I_1$. Then

$$||S_n^{(2k+2)}(f;t)||_{L_p(I_1)} \leq M \, n^{k+1}||f||_{L_p(I_1)}.$$

In addition, if f has $2k+2$ derivatives with $f^{(2k+1)} \varepsilon A.C.(I_1)$ and $f^{(2k+2)} \varepsilon L_p(I_1)$ then

$$||S_n^{(2k+2)}(f;t)||_{L_p(I_1)} \leq M_1 ||f^{(2k+2)}||_{L_p(I_1)},$$

where the constants M, M_1 are independent of n and f.

Lemma 4.2.13. Let $f \in L_p[A,B]$ $(1 \le p < \infty)$ with supp $f \subseteq I_1$. Then

$$\|\bar{S}_{n,m}^{(m+1)}(f,t)\|_{L_p(I_1)} \le M \, n^{\frac{m+1}{2}} \, \|f\|_{L_p(I_1)} \, ,$$

Moreover, if f has $m+1$ derivatives over I_1 with $f^{(m)} \in$ A.C. (I_1) and $f^{(m+1)} \in L_p(I_1)$ then

$$\|\bar{S}_{n,m}^{(m+1)}(f;t)\|_{L_p(I_1)} \le M_1 \|f^{(m+1)}\|_{L_p(I_1)}$$

where

$$\bar{S}_{n,m}(f;t) = \int_A^B W(n,t,u) \, \{ \sum_{j=0}^m \frac{n^{j/2}}{j!} \times$$

$$\times \, (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \, \Delta^j f(u) \} \, du.$$

Lemma 4.2.14. Let $1 \le p < \infty$, $h \in L_p[A,B]$ and $i,j \in \mathbb{N}^0$. Then for any fixed positive number ℓ ,

$$\|S_n(|u-t|^i \, |\int_t^u |u-w|^j \, |h(w)| \, dw|;t)\|_{L_p(I_2)}$$

$$\le M\{n^{-(i+j+1)/2} \, \|h\|_{L_p(I_1)} + n^{-\ell} \, \|h\|_{L_p(I)} \}.$$

Furthermore, if $-\infty < A < B < \infty$, then

$$\|S_n(|u-t|^i \, |\int_t^u |u-w|^j \, |h(w)| \, dw|;t)\|_{L_p[A,B]}$$

$$\le M_1 \, n^{-(i+j+1)/2} \, \|h\|_{L_p[A,B]} \, .$$

Lemma 4.2.15. Let $h \in B.V. (I_1)$ and $X(u)$ be the characteristic function of I_1. Then, for $i,j \in \mathbb{N}^0$, there holds

$$||S_n(X(u) |u-t|^i | \int_t^u |u-w|^j | dh(w) ||;t) ||_{L_1(I_2)}$$

$$\leq \frac{M_1}{n^{(i+j+1)/2}} ||h||_{B.V.(I_1)} .$$

Furthermore, if $A,B \in \mathbb{R}$ and $h \in B.V. (I)$, then

$$||S_n(|u-t|^i \int_t^u |u-w|^j| dh(w) ||;t) ||_{L_1(I)}$$

$$\leq M_2 \, n^{-(i+j+1)/2} ||h||_{B.V.(I)}$$

In above, M and M_1 are constants.

We combine the direct theorems of Sinha [61] for the operators $S_n(.,k,t)$ and $S_{n,m}(.,t)$ in the following

Theorem 4.2.16. Let $f \in L_p[A,B]$ $(1 \leq p < \infty)$. Then for all sufficiently large values of n,

1. $||S_n(f,k,t) - f(t)||_{L_p(I_2)}$

$$\leq M_1 \{\omega_{2k+2}(f,n^{-1/2},p,I_1) + n^{-(k+1)} ||f||_{L_p[A,B]} \}$$

and

2. $||S_{n,m}(f;t) - f(t)||_{L_p(I_2)}$

$$\leq M_2 \{\omega_{m+1}(f,n^{-1/2},p,I_1) + n^{-(m+2)/2} ||f||_{L_p(I)} \},$$

where $M_1; M_2$ are constants.

We close this section by stating inverse theorems of Sinha [61] , regarding $S_n(\cdot,k,t)$ and $S_{n,m}(\cdot,t)$.

<u>Theorem</u> 4.2.17. Let $0 < \alpha < 2k+2$ and $f \varepsilon L_p [A,B]$ $(1 \leq p <$ Then

$$||S_n(f,k,t) - f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}) \quad (n \to \infty)$$

implies that

$$\omega_{2k+2}(f,t,p,I_2) = O(t^\alpha) \quad (t \to 0).$$

<u>Theorem</u> 4.2.18. Let $0 < \alpha < m+1$ and $f \varepsilon L_r [A,B]$ $(1 \leq p < \infty)$. Then

$$||S_{n,m}(f;t) - f(t)||_{L_p(I_1)} = O(n^{-\alpha/2}) \quad (n \to \infty)$$

implies that

$$\omega_{m+1}(f,t,p,I_2) = O(t^\alpha) \quad (t \to 0).$$

4.3 $O(\varphi)$-inverse theorem for $S_n(\cdot,k,t)$.

Theorem 4.2.16 implies that, if $\omega_{2k+2}(f,t,p,I_1) = O(\varphi(t))$ $(t \to 0)$, then $S_n(\cdot,k,t) = O(\varphi(n^{-1/2}))$ $(n \to \infty)$. Now, in this section, we prove the corresponding $O(\varphi)$-inverse theorem.

<u>Theorem</u> 4.3.1. Let $\varphi \varepsilon \Phi_{2k+2}$ and $f \varepsilon L_p [A,B]$ $(1 \leq p < \infty)$. Then

(4.3.1) $||S_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2}))$ $(n \to \infty)$

implies that

$$(4.3.2) \qquad \omega_{2k+2}(f,t,p,I_2) = O(\varphi(t)) \quad (t \to 0).$$

<u>Proof.</u> Let (x_i,y_i), $i=1,2,3,4$ be such that $a_1 < x_1 < x_2 < x_3 < x_4 < a_2 < b_2 < y_4 < y_3 < y_2 < y_1 < b_1$.

<u>Case 1.</u> (For functions with a restricted support):
Suppose supp $f \subseteq (x_3,y_3)$. Then

$$||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3,y_3]} \leq ||\Delta_\gamma^{2k+2}\{f(t)-S_n(f,k,t)\}||_{L_p[x_3,}$$
$$+ ||\Delta_\gamma^{2k+2} S_n(f,k,t)||_{L_p[x_3,y_3]}.$$

Proceeding as in the proof of theorem 2.3.1, for sufficiently small η, $\gamma > 0$, we get

$$||\Delta_\gamma^{2k+2} S_n(f,k,t)||_{L_p[x_3,y_3]} \leq \gamma^{2k+2} ||S_n^{(2k+2)}(f,k,t)||_{L_p[x_2,}$$
$$\leq \gamma^{2k+2}\{S_n^{(2k+2)}(f-f_{\eta,2k+2},k,t)||_{L_p[x_2,y_2]}$$
$$+ ||S_n^{(2k+2)}(f_{\eta,2k+2},k,t)||_{L_p[x_2,y_2]}\}.$$

Hence, from lemmas 4.2.12 and 0.6.5 , we get

$$(4.3.3) \qquad ||\Delta_\gamma^{2k+2} S_n(f,k,t)||_{L_p[x_3,y_3]}$$
$$\leq M_1 \gamma^{2k+2}(n^{k+1}+ \frac{1}{\eta^{2k+2}}) \omega_{2k+2}(f,\eta,p, [x_3,y_3]).$$

Now, since supp $f \subseteq (x_3,y_3)$ and γ is sufficiently small, by hypothesis we get that

$$(4.3.4) \quad ||\Delta_\gamma^{2k+2}\{S_n(f,k,t)-f(t)\}||_{L_p[x_3,y_3]}= O(\varphi(n^{-1/2})) \ (n \to \infty$$

Thus, from (4.3.3-4), we have

$$||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3,Y_3]} \leq M_2\{\varphi(n^{-1/2}) + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}})$$

$$\times\ \omega_{2k+2}(f,\eta,p,[x_3,Y_3])\}.$$

Choosing n such that $n \leq \eta^{-2} < 2n$, we observe that $\varphi(n^{-1/2}) \leq K_{2^{-1/2}}\varphi(\eta)$ and hence

$$||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3,Y_3]}$$

$$\leq M_3\{\varphi(\eta) + (\frac{\gamma}{\eta})^{2k+2}\ \omega_{2k+2}(f,\eta,p,[x_3,Y_3])\}.$$

This being true for all γ sufficiently small, taking supremum over all $\gamma \leq t$, we get

$$\omega_{2k+2}(f,t,p,[x_3,Y_3])$$

$$\leq M_4\{\varphi(\eta) + (\frac{t}{\eta})^{2k+2}\ \omega_{2k+2}(f,\eta,p,[x_3,Y_3])\},$$

which by lemma 1.2.2 implies that

$$\omega_{2k+2}(f,t,p,[x_3,Y_3]) = O(\varphi(t)) \quad (t \to 0).$$

The conclusion, now, follows since $I_2 \subset (x_3,Y_3)$.

Case 2: (The general case) Now, we prove the theorem for general f.

Let $g \in C_0^{2k+2}$ such that supp $g \subset (x_3,Y_3)$ and $g(t) = 1$ for $t \in [x_4,Y_4]$.

In view of the case 1, it suffices to prove that

(4.3.5) $\|fg(t) - S_n(fg,k,t)\|_{L_p[x_3,y_3]} = O(\varphi(n^{-1/2}))$ $(n \to \infty)$.

Let $\varphi \in \Phi_1$. Then, for some ξ lying between u and t,

$$\|S_n(fg,k,t) - (fg)(t)\|_{L_p[x_3,y_3]}$$

$$\leq \|S_n((f(u)-f(t))g(t),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|S_n(f(u)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}.$$

$$\leq M_4 \varphi(n^{-1/2}) + \|S_n(f(u)(u-t)g'(\xi),k,t)\|_{L_p[x_3,y_3]}.$$

Thus, from hypothesis (4.3.1) and lemma 4.2.11, we get

$$\|S_n(fg,k,t)-(fg)(t)\|_{L_p[x_3,y_3]}$$

$$\leq M_5\{\varphi(n^{-1/2}) + n^{-1/2}\}$$

$$\leq M_6 \varphi(n^{-1/2}),$$

since $\varphi \in \Phi_1$. Thus, the theorem is proved for all $\varphi \in \Phi_1$.

Also, we observe that $\varphi_1 \subset \Phi_2 \subset \cdots \subset \Phi_{2k+2}$. Hence, to complete the proof of the theorem, it is sufficient to prove (4.3.5) for all $\varphi \in \Phi_{r+1}$, assuming the theorem for all $\varphi \in \Phi_r$ where $r \in \mathbb{N}$ is such that $1 \leq r \leq 2k+1$. Let $\varphi \in \Phi_{r+1}$. We have

$$\|S_n(fg,k,t) - (fg)(t)\|_{L_p[x_3,y_3]}$$

$$\leq \|S_n((f(u)-f(t))g(t),k,t)\|_{L_p[x_3,y_3]} +$$

$$+ \|S_n(f(u)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$\leq M_7\, \varphi(n^{-1/2}) + \|S_n(f(u)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$\leq M_7\, \varphi(n^{-1/2}) +$$

$$+ \|S_n((f(u)-f_{\eta,2k+2}(u))(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|S_n((f_{\eta,2k+2}(u)-f_{\eta,2k+2}(t))(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$+ \|S_n(f_{\eta,2k+2}(t)(g(u)-g(t)),k,t)\|_{L_p[x_3,y_3]}$$

$$(4.3.6) = M_7\, \varphi(n^{-1/2}) + Z_1+Z_2+Z_3, \text{ say.}$$

By lemmas 4.2.9 and 0.6.5

$$(4.3.7) \quad Z_3 \leq \frac{M_8}{n^{k+1}}\, \|f\|_{L_p(I)}\,.$$

For some ξ lying between u and t,

$$Z_1 = \|S_n((f(u)-f_{\eta,2k+2}(u))(u-t)g'(\xi),k,t)\|_{L_p[x_3,y_3]}\,.$$

Now, using the definition of the linear combinations and lemmas 4.2.11 and 0.6.5 (taking $\ell = k+1$), we get

$$(4.3.8) \quad Z_1 \leq M_9\{n^{-1/2}\,\omega_{2k+2}(f,\eta,p,[x_1,y_1]) + n^{-(k+1)}\|f\|_{L_p(I)}\}$$

For some ξ lying between u and t,

$$Z_2 \leq \sum_{i=1}^{2k+1}\ \sum_{j=1}^{2k}\ \frac{1}{i!j!}\|f_{\eta,2k+2}^{(i)}(t)g^{(j)}(t)S_n((u-t)^{i+j},k,t)\|_{L_p[x_3,y_3]}+$$

$$+ \frac{1}{(2k+1)!} \left\{ \sum_{i=1}^{2k+1} \frac{1}{i!} \left\| S_n(f_{\eta,2k+2}^{(i)}(t) g^{(2k+1)}(\xi) \times \right. \right.$$

$$\times (u-t)^{2k+1+i}, k, t) \Big\|_{L_p[x_3, y_3]}$$

$$+ \frac{1}{(2k+1)!} \left\| S_n\left(\left(\sum_{i=1}^{2k} \frac{g^{(i)}(t)}{i!} (u-t)^i \right) \times \right. \right.$$

$$\times \left(\int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\,dw \right), k, t) \Big\|_{L_p[x_3, y_3]}$$

$$+ \frac{1}{((2k+1)!)^2} \left\| S_n((u-t)^{2k+1} g^{(2k+1)}(\xi) \times \right.$$

$$\times \int_t^u (u-w)^{2k+1} f_{\eta,2k+2}^{(2k+2)}(w)\,dw, k, t) \Big\|_{L_p[x_3, y_3]}$$

$$(4.3.9) \qquad = R_1 + R_2 + R_3 + R_4, \text{ say.}$$

Using lemmas 4.2.14–15, (taking $\ell = 2k+2$), we have

$$(4.3.10) \quad R_3 \leq M_{10}\{ n^{-(k+3/2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p[x_2, y_2]}$$

$$+ n^{-(2k+2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p(I)} \}$$

and

$$(4.3.11) \quad R_4 \leq M_{11}\{ n^{-(2k+3/2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p[x_2, y_2]}$$

$$+ n^{-(2k+2)} \| f_{\eta,2k+2}^{(2k+2)} \|_{L_p(I)} \}.$$

Since $\sum_{j=0}^{k} C(j,k) d_j^{-m} = 0$, $m = 1, 2, \ldots, k$, from lemma 4.2.2, it follows that

$$R_1 \leq \frac{M_{12}}{n^{k+1}} \left(\sum_{i=1}^{2k+1} ||f_{\eta,2k+2}^{(i)}||_{L_p[x_3,y_3]} \right)$$

and by corollary 4.2.3, we get

$$R_2 \leq \frac{M_{13}}{n^{k+1}} \left(\sum_{i=1}^{2k+1} ||f_{\eta,2k+2}^{(i)}||_{L_p[x_3,y_3]} \right).$$

Now applying lemma C.6.3 , we get

$$(4.3.12) \quad R_1, R_2 \leq \frac{M_{14}}{n^{k+1}} \{ ||f_{\eta,2k+2}^{(2k+1)}||_{L_p[x_3,y_3]} + ||f_{\eta,2k+2}||_{L_p[x_3,y_3]} \}.$$

Choosing n such that $n \leq n^{-2} < 2n$, from (4.3.7-12) and lemma C.6.5

$$(4.3.13) \quad Z_1 + Z_2 + Z_3 \leq M_{15} \{ n^{-1/2} \omega_{2k+2}(f, n^{-1/2}, p, [x_1, y_1])$$

$$+ n^{-1/2} \omega_{2k+1}(f, n^{-1/2}, p, [x_1, y_1]) + n^{-(k+1)} ||f||_{L_p(I)} \}$$

We can observe that, since $\varphi \in \Phi_{r+1}$,

$$\varphi^*(t) = \frac{\varphi(t)}{t} \in \Phi_r .$$

Now, since the theorem is true for all $\varphi \in \Phi_r$, we have

$$(4.3.14) \quad \omega_{2k+2}(f, t, p, [x_1, y_1]) = O(\varphi^*(t)) \quad (t \to 0).$$

Also, lemma 1.2.4 and the fact that $\varphi^* \in \Phi_{2k+1}$ imply that

$$(4.3.15) \quad \omega_{2k+1}(f, t, p, [x_1, y_1]) = O(\varphi^*(t)) \quad (t \to 0).$$

Thus, (4.3.5) follows from (4.3.13-15) and the theorem is proved.

Corollary 4.3.2. Let $1 \leq m \leq 2k+2$, $\varphi \varepsilon \Phi_m$ and $f \varepsilon L_p(I)(1 \leq p < \infty)$. Then

$$||S_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_m(f,t,p,I_2) = O(\varphi(t)) \quad (t \to 0).$$

Corollary 4.3.3. Theorem 4.2.17.

4.4 $o(\varphi)$-inverse theorem for $S_n(.,k,t)$

It follows from theorem 4.2.16 that, if $\omega_{2k+2}(f,t,p,I_1) = o(\varphi(t)) (t \to 0)$ then $||S_n(f,k,t) - f(t)||_{L_p(I_2)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$. The $o(\varphi)$-inverse theorem for the operator $S_n(.,k,t)$ is as follows:

Theorem 4.4.1. Let $f \varepsilon L_p(I)$ $(1 \leq p < \infty)$ and $\varphi \varepsilon \Phi_{2k+2}$. Then

$$(4.4.1) \quad ||S_n(f,k,t) - f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$(4.4.2) \quad \omega_{2k+2}(f,t,p,I_2) = o(\varphi(t)).$$

Proof. Let x_i, y_i be such that, for $i=1,2,3$,
$a_1 < x_i < x_i+1 < a_2 < b_2 < y_{i+1} < y_i < b_1.$

<u>Case 1</u>. (For functions with restricted support).

Suppose supp $f \subset (x_3, y_3)$. Then proceeding as in the proof of theorem 4.3.1, for sufficiently small $\gamma, \eta > 0$, we get

$$(4.4.3) \quad ||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3, y_3]} \leq M_1 \{\varphi(n^{-1/2}) +$$

$$+ \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \, \omega_{2k+2}(f, \eta, p, [x_3, y_3])\}.$$

Now, we define $\psi(x)$ as follows:

$$\psi(x) = \left\{ \begin{array}{l} ||\Delta_\gamma^{2k+2}\{f(t) - S_n(f, k, t)\}||_{L_p[x_3, y_3]} \quad \text{if} \quad x = n^{-1/2} \\ \psi(n^{-1/2}) \qquad \text{if} \quad x \, \varepsilon \, ((n+1)^{-1/2}, n^{-1/2}) \end{array} \right.$$

for $n = 1, 2, \ldots$.

Since γ is sufficiently small and supp $f \subset (x_3, y_3)$, by hypothesis (4.4.1), we get that

$$(4.4.4) \quad ||\Delta_\gamma^{2k+2}\{f(t) - S_n(f, k, t)\}||_{L_p[x_3, y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Also, if $t \, \varepsilon \, ((n+1)^{-1/2}, n^{-1/2})$ then

$$\frac{\psi(t)}{\varphi(t)} \leq K_2 \frac{\psi(n^{-1/2})}{\varphi(n^{-1/2})}$$

and hence $\psi(t) = o(\varphi(t)) \quad (t \to 0)$.

Thus, from (4.4.3-4), we get

$$||\Delta_\gamma^{2k+2} f(t)||_{L_p[x_3, y_3]} \leq M_2 \{\psi(n^{-1/2}) + \gamma^{2k+2}(n^{k+1} + \frac{1}{\eta^{2k+2}}) \times$$

$$\times \, \omega_{2k+2}(f, \eta, p, [x_3, y_3])\}.$$

Taking supremum over $\gamma \leq t$ and choosing n such that $n \leq \eta^{-2} < n+1$, we get

$$\omega_{2k+2}(f,t,p,[x_3,y_3])$$

$$\leq M_3\{\psi(\eta) + (t/\eta)^{2k+2}\omega_{2k+2}(f,\eta,p,[x_3,y_3])\},$$

which in view of lemma 1.2.3 proves the theorem for case 1.

Case 2. (general functions).

Let $g \in C_o^{2k+2}$ such that supp $g \subset (x_3,y_3)$ and $g(t) = 1$ for $t \in [x_4,y_4]$.

Again, proceeding as in the proof of theorem 4.3.1, we are left to show that

$$(4.4.5) \quad ||fg(t) - S_n(fg,k,t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Let $\varphi \in \Phi_1$. Then, clearly as for theorem 4.3.1, it follows that

$$||(fg)(t)-S_n(fg,k,t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

and hence the theorem is proved for all $\varphi \in \Phi_1$. Next, assume that the result holds for all $\varphi \in \Phi_r$ $(1 \leq r \leq 2k+1)$ and let $\varphi \in \Phi_{r+1}$. Again proceeding as in theorem 4.3.1, we get

$$||S_n((fg),k,t) - (fg)(t)||_{L_p[x_3,y_3]} = Z_0 + Z_1 + Z_2 + Z_3,$$

where

$$Z_o = ||S_n((f(u)-f(t))g(t),k,t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty),$$

Z_1, Z_2, Z_3 are as before and satisfy

$$(4.4.6) \quad Z_1+Z_2+Z_3 \leq M_4\{n^{-1/2} \, \omega_{2k+2}(f,n^{-1/2},p,[x_1,y_1])$$

$$+ n^{-1/2} \, \omega_{2k+1}(f,n^{-1/2},p,[x_1,y_1]) + n^{-(k+1)}||f||_{L_p(I)}\}$$

Now, defining $\varphi^*(t) = \varphi(t)/t$, we observe that $\varphi^* \varepsilon \, \Phi_r$. Hence by the assumption, we get

$$(4.4.7) \quad \omega_{2k+2}(f,t,p,[x_1,y_1]) = o(\varphi^*(t)) \quad (t \to 0),$$

which from lemma 1.2.5 and the fact that $\varphi^* \varepsilon \, \Phi_{2k+1}$ implies that

$$(4.4.8) \quad \omega_{2k+1}(f,t,p,[x_1,y_1]) = o(\varphi^*(t)) \quad (t \to 0).$$

Thus, from (4.4.6-4.4.8), we get (4.4.5) and hence the theorem for all $\varphi \, \varepsilon \, \Phi_{r+1}$.

Thus, we have proved the theorem, first, for all $\varphi \, \varepsilon \, \Phi_1$. Later, assuming the theorem for $\varphi \, \varepsilon \, \Phi_r$ $(1 \leq r \leq 2k+1)$, we proved it for $\varphi \, \varepsilon \, \Phi_{r+1}$.

The theorem, now, follows for all $\varphi \, \varepsilon \, \Phi_{2k+2}$, since $\Phi_1 \subset \Phi_2 \subset \cdots \subset \Phi_{2k+2}$.

As in section 4.3, we get an immediate

Corollary 4.4.2. Let $f \, \varepsilon \, L_p(I)$ and $\varphi \, \varepsilon \, \Phi_r$ for some $1 \leq r \leq 2k+2$. Then

$$||S_n(f,k,t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = o(\varphi(t)) \quad (t \to 0).$$

4.5 $O(\varphi)$-inverse theorem for $S_{n,m}(\cdot,t)$

Theorem 4.2.16 implies that if

$$\omega_{m+1}(f,t,p,I_1) = O(\varphi(t)) \quad (t \to 0) \quad \text{then}$$

$$||S_{n,m}(f;t) - f(t)||_{L_p(I_2)} = O(\varphi(n^{-1/2})) \quad (n \to \infty).$$

Now, the corresponding inverse theorem is as follows:

Theorem 4.5.1. Let $f \varepsilon L_p(I)$ and $\varphi \varepsilon \Phi_{m+1}$. Then

$$||S_{n,m}(f;t) - f(t)|| = O(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_{m+1}(f,t,p,I_2) = O(\varphi(t)) \quad (t \to 0).$$

Proof. Let x_i, y_i, i=1,2,3,4 satisfy

$$a_1 < x_1 < x_2 < x_3 < x_4 < a_2 < b_2 < y_4 < y_3 < y_2 < y_1 < b_1.$$

Suppose supp $f \subseteq (x_3,y_3)$. Defining

$$F(t,u) = \sum_{j=0}^{m} \{\frac{n^{j/2}}{j!} (\prod_{i=0}^{j-1} (t-u-\frac{i}{n^{1/2}})) \Delta^j f(u)\}$$

and

$$\bar{S}_{n,m}(f;t) = \int_A^B w(n,t,u) F(t,u)du,$$

we observe that

$$S_{n,m}(f;t) = \bar{S}_{n,m}(f;t) - \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} \times$$

$$\times F(t,u) du.$$

Hence, for sufficiently small values of γ, we have

$$||\Delta_\gamma^{m+1} f(t'||_{L_p[x_3,y_3]} \leq ||\Delta_\gamma^{m+1}\{f(t'-S_{n,m}(f;t)\}||_{L_p[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{m+1} S_{n,m}(f;t)||_{L_p[x_3,y_3]}$$

$$\leq ||\Delta_\gamma^{m+1}\{f(t)-S_{n,m}(f;t)\}||_{L_p[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{m+1} \bar{S}_{n,m}(f;t)||_{L_p[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{m+1}\{\int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}}|u-t|^{m_o+2} F(t,u) du\}||_{L_p[x_3,y_3]}$$

$$(4.5.1) \qquad = J_1+J_2+J_3, \quad \text{say}.$$

Proceeding as in theorem 2.3.1 and using lemmas 4.2.13 and 0.6.5, we get

$$(4.5.2) \quad J_2 \leq M_2 \gamma^{m+1}(n^{\frac{m+1}{2}} + \eta^{-(m+1)}) \omega_{m+1}(f,\eta,p,[x_2,y_2]).$$

To obtain an estimate for J_3, consider

$$T_1(t) = cn^{\frac{(j-r)}{2}} \int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_o+2}} |u-t|^{m_o+2} (t-u)^{j-r}(\Delta^j f(u)) du,$$

where $0 \leq r \leq j-1$ and c is a constant.

Since supp $f \subset (x_3, y_3)$, lemma 4.2.11 implies that

$$||T_1(t)||_{L_p[x_3, y_3+(m+1)\gamma]} \leq \frac{M_3}{n^{(m+2)/2}} ||f||_{L_p[x_3, y_3]}$$

Consequently,

(4.5.3) $$J_3 \leq \frac{M_4}{n^{(m+2)/2}} ||f||_{L_p[x_3, y_3]} \cdot$$

Thus, from (4.5.1-3), we get

$$||\Delta_\gamma^{m+1} S_{n,m}(f; t)||_{L_p[x_3, y_3]}$$

$$\leq M_5 \; \gamma^{m+1} \{(n^{(\frac{m+1}{2})} + \eta^{-(m+1)}) \; \omega_{m+1}(f, \eta, p, [x_2, y_2])$$

$$+ n^{-(m+2)/2} ||f||_{L_p[x_3, y_3]} \}.$$

Again, proceeding as in theorem 4.3.1, we get the result for all functions f with supp $f \subset (x_3, y_3)$. Also in the case of general support, we are left to show that

(4.5.4) $$||(fg)(t) - S_{n,m}(fg; t)||_{L_p[x_3, y_3]} = 0(\varphi(n^{-1/2})) \; (n \to \infty),$$

where $g \varepsilon C_o^{m+1}$ such that supp $g \subset (x_3, y_3)$ and $g(t) = 1$ on $[x_4, y_4]$.

Now, we prove (4.5.4) by induction as follows:

First, we prove the theorem for all $\varphi \varepsilon \Phi_1$ and next assuming the theorem for all $\varphi \varepsilon \Phi_r$ $(1 \leq r \leq m)$, we prove the theorem for all $\varphi \varepsilon \Phi_{r+1}$.

Let $\varphi \in \Phi_1$. Then, for some ξ lying between u and t,

$$||S_{n,m}((fg);t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq M_6 \, \varphi(n^{-1/2}) + ||S_{n,m}(f(u)(u-t)g'(\xi);t)||_{L_p[x_3,y_3]}.$$

Now, using lemma 4.2.11 and the fact that $t = o(\varphi(t))$, we get (4.5.3) and hence the theorem is proved for all $\varphi \in \Phi_1$.

Now, assume that the theorem holds for all $\varphi \in \Phi_r$ $(1 \leq r \leq 2k+1)$. Let $\varphi \in \Phi_{r+1}$. Now

$$||S_{n,m}(fg;t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$\leq M_7 \, \varphi(n^{-1/2}) + ||S_{n,m}((f(u)-f_{\eta,m+1}(u)) \times$$

$$\times (g(u)-g(t));t)||_{L_p[x_3,y_3]}$$

$$+ ||S_{n,m}((f_{\eta,m+1}(u) - f_{\eta,m+1}(t))(g(u)-g(t));t)||_{L_p[x_3,y_3]}$$

$$+ ||S_{n,m}((f_{\eta,m+1}(t))(g(u)-g(t));t)||_{L_p[x_3,y_3]}$$

$$= M_7 \, \varphi(n^{-1/2}) + Z_1 + Z_2 + Z_3, \text{ say.}$$

For some ξ lying between u and t,

$$Z_1 \leq ||S_{n,m}((f(u)-f_{\eta,m+1}(u))(u-t)g'(\xi);t)||_{L_p[x_3,y_3]}$$

Now, using lemmas 4.2.11 and 0.6.5, we get

$$(4.5.5) \quad Z_1 \leq M_8 \{n^{-1/2}\omega_{m+1}(f,\eta,p,[x_1,y_1]) + n^{-(m+1)/2}||f||_{L_p(I)}\}$$

Also, by theorem 4.2.10 and lemma 0.6.5

$$(4.5.6) \quad Z_3 \leq \frac{M_9}{n^{(m+1)/2}} ||f||_{L_p(I)} .$$

Using Taylor expansions of $f_{\eta,m+1}$ and g we can write
$Z_2 \leq \sum_{i=1}^{4} R_i$, where R_1, R_2, R_3, R_4 are as in theorem 2.3.1
with $P_{n,r}$ replaced by $S_{n,m}$.

Consider

$$T_2(t) = c \, f_{\eta,m+1}^{(i)}(t) \, n^{(j-r-s)/2}$$

$$\int_A^B \frac{W(n,t,u)}{1+|u-t|^{m_0+2}} (u-t)^{i+j+m-r-s} g^{(m)}(\xi_j) du,$$

where $1 \leq i \leq m$, $0 \leq j \leq m$, $0 \leq r \leq j-1$, $0 \leq s \leq m+i$, ξ_j
lies between $u + \frac{i}{n^{1/2}}$ and t and c is a constant.

Applying corollary 4.2.3,

$$|T_2(t)| \leq \frac{M_{10}}{n^{(m+i)/2}} |f_{\eta,m+1}^{(i)}(t)| .$$

As, $T_2(t)$ is a typical component in R_2,

$$(4.5.7) \quad R_2 \leq M_{11} \{ \sum_{i=1}^{m} n^{-(m+i)/2} ||f_{\eta,m+1}^{(i)}||_{L_p[x_3,y_3]} \} .$$

Now, from lemmas 4.2.4 and 4.2.2

$$(4.5.8) \quad R_1 \leq M_{12} \sum_{\substack{i=1 \\ i+j>m}}^{m} \sum_{j=1}^{m-1} n^{-(i+j)/2} ||f_{\eta,m+1}^{(i)}||_{L_p[x_3,y_3]} .$$

A typical component of

$$S_{n,m}(g^{(m)}(\xi)(u-t)^k\{\int_t^u (u-w)^m f_{\eta,m+1}^{(m+1)}(w)\,dw\};t)$$

is of the form

$$T_3(t) = c\, n^{(\theta-k)/2}\, \int_A^B \psi(t,u)\,du$$

where

$$\psi(t,u) = \frac{W(n,t,u)}{1+|u-t|^{m_o+2}}\,(u-t)^\theta\, g^{(m)}(\xi_{r_2})\, \times$$

$$\times \int_t^{u+\frac{r_2}{n^{1/2}}} (u-w+\frac{r_2}{n^{1/2}})^m\, f_{\eta,m+1}^{(m+1)}(w)\,dw$$

and $\theta = j + r_j - r_1$, $0 \le j \le m$, $0 \le r_1 \le j-1$, $0 \le r_2 \le j$, $0 \le r_3 \le k$, ξ_{r_2} lies between $u+\frac{r_2}{n^{1/2}}$ and t and c is a constant.

Let $X(u)$ be the characteristic function of $[x_2, y_2]$. Then

$$T_3(t) = T_{31}(t) + T_{32}(t), \text{ say, where}$$

$$T_{31}(t) = c\, n^{(\theta-k)/2}\, \int_A^B X(u)\, \psi(t,u)\,du$$

and

$$T_{32}(t) = c\, n^{(\theta-k)/2}\, \int_A^B (1-X(u))\, \psi(t,u)\,du.$$

It follows from lemmas 4.2.14-15 that

$$||T_{31}(t)||_{L_p[x_3,y_3]} \leq \frac{M_{13}}{n^{(m+k+1)/2}} ||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]} .$$

It is easy to see that, for any $\ell > 0$

$$||T_{32}(t)||_{L_p[x_3,y_3]} \leq \frac{M_{14}}{n^\ell} ||f^{(m+1)}_{\eta,m+1}||_{L_p(I)} .$$

Thus,

$$||S_{n,m}(g^m(\xi)(u-t)^k(\int_t^u (u-w)^m f^{(m+1)}_{\eta,m+1}(w)dw);t)||_{L_p[x_3,y_3]}$$

$$\leq M_{15}\{n^{-(m+k+1)/2}||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]}$$

$$+ n^{-\ell}||f^{(m+1)}_{\eta,m+1}||_{L_p(I)}\}$$

and hence

$$(4.5.9) \quad R_3 \leq M_{16}\{n^{-(m+2)/2}||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]}$$

$$+ n^{-\ell}||f^{(m+1)}_{\eta,m+1}||_{L_p(I)}\}$$

and

$$(4.5.10) \quad R_4 \leq M_{17}\{n^{-(2m+1)/2}||f^{(m+1)}_{\eta,m+1}||_{L_p[x_2,y_2]}$$

$$+ n^{-\ell}||f^{(m+1)}_{\eta,m+1}||_{L_p(I)}\}$$

Taking $\ell = m+1$ and applying lemma 0.6.5 and combining (4.5.7-10), we get

$$(4.5.11) \quad Z_2 \leq M_{18}\{n^{-(m+2)/2}||f_{\eta,m+1}^{(m+1)}||_{L_p[x_2,y_2]}$$

$$+ n^{-(m+1)/2}||f_{\eta,m+1}^{(m)}||_{L_p[x_3,y_3]}$$

$$+ n^{-(m+1)/2}||f_{\eta,m+1}||_{L_p[x_3,y_3]}$$

$$+ n^{-(m+1)}||f_{\eta,m+1}^{(m+1)}||_{L_p(I)}\}.$$

Applying lemma 0.6.5 and choosing n such that $n \leq \eta^{-2} < 2n$, from (4.5.5-6,11), we get

$$Z_1+Z_2+Z_3 \leq M_{19}\{n^{-1/2}\omega_{m+1}(f,n^{-1/2},p,[x_1,y_1])$$

$$+ n^{-1/2}\omega_m(f,n^{-1/2},p,[x_1,y_1]) + n^{-(m+1)/2}||f||_{L_p(I)}\}.$$

After this step, the rest of the proof is as in the case of theorem 4.3.1.

Corollary 4.5.2. Let $f \in L_p(I)$ and $\varphi \in \Phi_k$ for some k such that $1 \leq k \leq m+1$. Then

$$||S_{n,m}(f;t) - f(t)||_{L_p(I_1)} = O(\varphi(n^{-1/2})) \qquad (n \to \infty)$$

implies that

$$\omega_k(f,t,p,I_2) = O(\varphi(t)) \quad (t \to 0).$$

Taking $\varphi(t) = t^\alpha$ $(0 < \alpha < m+1)$, we get

Corollary 4.5.3. Theorem 4.2.18.

4.6 $o(\varphi)$-inverse theorem for $S_{n,m}(.,t)$

In this section, we prove $o(\varphi)$-inverse theorem for the operator $S_{n,m}(.,t)$ whose $o(\varphi)$-direct theorem is a consequence of Theorem 4.2.18.

<u>Theorem</u> 4.6.1.   Let $f \subset L_p(I)$ and $\varphi \in \Phi_{m+1}$.   Then

$$(4.6.1)   \|S_{n,m}(f,t)-f(t)\|_{L_p(I_1)}= o(\varphi(n^{-1/2}))   (n \to \omega)$$

implies that

$$(4.6.2)   \omega_{m+1}(f,t,p,I_2) = o(\varphi(t))   (t \to 0).$$

Proof.   Let x_i, y_i be as in the proof of theorem 4.5.1. Suppose supp $f \subset (X_3, y_3)$ from (4.5.1-3) we get

$$\|\Delta_\gamma^{m+1}S_{n,m}(f,t)\|_{L_p[x_3,y_3]}$$

$$\leq M_1 \gamma^{m+1} \{(n^{(\frac{m+1}{2})} + \eta^{-(m+1)})\omega_{m+1}(f,\eta,p,[x_3,y_3])$$

$$+ n^{-(\frac{m+2}{2})} \|f\|_{L_p[x_3,y_3]} \}$$

Using the fact that $t^{m+1}/\varphi(t) \to 0$ as $t \to 0$, we have

$$(4.6.3)   \|\Delta_\gamma^{m+1}S_{n,m}(f,t)\|_{L_p[x_3,y_3]} \leq M_2 \{\gamma^{m+1}(n^{(\frac{m+1}{2})}+\eta^{-(m+1)}) \times$$

$$\times \omega_{m+1}(f,\eta,p,[x_3,y_3]) + \varphi(n^{-1/2})\}$$

By hypothesis (4.6.1) and since supp f $\subseteq$ (x_3,y_3), choosing γ sufficiently small, we get

(4.6.4) $||\Delta_\gamma^{2k+2}\{f(t)-S_{n,m}(f,k,t)\}||_{L_p}[x_3,y_3] = o(\varphi(n^{-1/2})) \ (n\to\infty).$

Now defining

$$\psi(x) = \begin{cases} ||\Delta_\gamma^{2n+1}\{f(t)-S_{n,m}(f;t)\}||_{L_p}[x_3,y_3] & \text{if } x = n^{-1/2} \\ \psi(n^{-1/2}) & \text{if } x\varepsilon((n+1)^{-1/2},n^{-1/2}), \end{cases}$$

for $n=1,2,\ldots$, we observe that

$$\psi(x) = o(\varphi(x)) \qquad (x\to 0).$$

Now, combining (4.6.3-4) and choosing n such that $n \le \eta^{-2} < n+1$, we get

$$||\Delta_\gamma^{m+1}f(t)||_{L_p}[x_3,y_3] \le M_3\{\psi(\eta) + (\tfrac{\gamma}{\eta})^{m+1}\omega_{m+1}(f,\eta,p,[x_3,y_3])\}.$$

Now, taking supremum over $\gamma \le t$, we get

$$\omega_{m+1}(f,t,p,[x_3,y_3]) \le M_4\{\psi(\eta) + (\tfrac{t}{\eta})^{m+1} \times$$

$$\times \ \omega_{m+1}(f,\eta,p,[x_3,y_3]\},$$

which proves the result by lemma 1.2.3. Thus, the theorem is proved for all f such that

$$\text{supp } f \subset (x_3,y_3).$$

Now we prove the theorem for general functions. We prove this by induction as follows: First, we prove the theorem for all $\varphi \varepsilon \Phi_1$. Next assuming the result of the theorem for all $\varphi \varepsilon \Phi_r$ for some r such that $1 \le r \le m$, we prove it for all $\varphi \varepsilon \Phi_{r+1}$.

Let $g \in C_o^{m+1}$ be such that supp $g \subset (x_3, y_3)$ and $g(t) = 1$ on $[x_4, y_4]$. If $\varphi \in \Phi_1$ then

$$||S_{n,m}(fg;t) - (fg)(t)||_{L_p[x_3,y_3]}$$

$$= ||S_{n,m}((f(u) - f(t))g(t); t)||_{L_p[x_3,y_3]}$$

$$+ ||S_{n,m}(f(u)(g(u) - g(t)); t)||_{L_p[x_3,y_3]}$$

$$= ||g(t)\{S_{n,m}(f;t) - f(t)\}||_{L_p[x_3,y_3]}$$

$$+ ||S_{s,n}(f(u)(u-t)g'(\xi), t)||_{L_p[x_3,y_3]}$$

for some ξ lying between u and t.

Thus, from lemma 4.2.11 and using the fact that $t = o(\varphi(t))$, we get

$$||S_{n,m}(fg;t) - (fg)(t)||_{L_p[x_3,y_3]} = o(\varphi(n^{-1/2})) \quad (n \to \infty).$$

The result follows, now for all $\varphi \in \Phi_1$, since supp $fg \subset (x_3,y_3)$ and $fg = f$ on I_2.

Next, assume that the theorem holds for all $\varphi \in \Phi_r$ for some r such that $1 \le r \le 2k+1$ and let $\varphi \in \Phi_{r+1}$. Then, from the proof of theorem 4.5.1, we get

$$||S_{n,m}(fg;t)-(fg)(t)||_{L_p[x_3,y_3]}$$

$$= ||g(t)\{S_{n,m}(f;t)-f(t)\}||_{L_p[x_3,y_3]}$$

$$+ ||S_{n,m}(f(u)(g(u)-g(t));t)||_{L_p[x_3,y_3]}$$

$$= o(\varphi(n^{-1/2}) + Z_1+Z_2+Z_3 ,$$

where $Z_1+Z_2+Z_3$ is estimated as

$$(4.6.5) \quad Z_1+Z_2+Z_3 \leq M\{n^{-1/2} \omega_{m+1}(f,n^{-1/2},p[x_1,y_1])$$

$$+ n^{-1/2} \omega_m (f,n^{-1/2},p[x_1,y_1]) + n^{-(\frac{m+1}{2})} ||f||_{L_p(I)}\}$$

It is easy to see that $\varphi^* = \dfrac{\varphi(t)}{t} \varepsilon \Phi_r$. Hence, by induction hypothesis, we get

$$(4.6.6) \quad \omega_{m+1}(f,n^{-1/2},p,[x_1,y_1]) = o(\varphi^*(n^{-1/2}))(n \to \infty)$$

which coupled with lemma 1.2.5 implies that

$$(4.6.7) \quad \omega_m(f,n^{-1/2},p,[x_1,y_1]) = o(\varphi^*(n^{-1/2})) \quad (n \to \infty).$$

Combining (4.6.5-7) with the fact that $n^{-(\frac{m+1}{2})}/\varphi(n^{-1/2}) \to 0$ as $n \to \infty$, the result follows .

Corollary 4.6.2. Let $f \varepsilon L_p[A,B]$ $(1 \leq p < \infty)$ and $\varphi \varepsilon \Phi_r$ $(1 \leq r \leq m+1)$. Then

$$||S_{n,m}(f;t)-f(t)||_{L_p(I_1)} = o(\varphi(n^{-1/2})) \quad (n \to \infty)$$

implies that

$$\omega_r(f,t,p,I_2) = o(\varphi(t)) \quad (t \to 0)$$

CHAPTER V

φ-INVERSE THEOREMS FOR LINEAR COMBINATIONS OF GENERALISED JACKSON OPERATORS

5.1 Introduction.

In this chapter we study the L_q-approximation $(1 \leq q < \infty)$ by linear combinations $L_{n,p}(.,k,t)$ of the generalised Jackson operators $L_{n,p}(.,t)$. Generalised Jackson's operators $L_{n,p}$ have been studied by Schurer [56], Schurer and Steutel [57], Lorentz [36] Rathore [49,51] and others.

For any $f \in L_{q,2\pi}$ $(1 \leq q < \infty)$, the class of 2π-periodic functions on $\mathbb{R}$ whose restriction on $[-\pi,\pi]$ belongs to $L_q [-\pi,\pi]$, the operator $L_{n,p}(f,t)$ $(n,p \in \mathbb{N})$ is defined by

$$(5.1.1) \quad L_{n,p}(f;t) = \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin n(\frac{u-t}{2})}{\sin (\frac{u-t}{2})}\right)^{2p} f(u) \, du,$$

where

$$(5.1.2) \quad A_{n,p} = \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} du.$$

The linear combinations $L_n(.,k,t)$ of $L_n(.,t)$ are defined as follows :

Let $d_0,...,d_k$ be $k+1$ distinct positive integers. Then

$$(5.1.3) \quad L_{n,p}(.,k,t) = \sum_{j=0}^{k} C(j,k) L_{d_j n,p}(.,t)$$

where

$$(5.1.4) \quad C(j,k) = \begin{cases} \displaystyle\sum_{\substack{i=0 \\ i\neq j}}^{k} \frac{d_j}{d_j-d_i}, & \text{if } k \neq 0 \\ 1, & \text{if } k = 0 \end{cases}$$

Schurer [56] proved the uniform convergence of the sequence $\{L_{n,p}(.,t)\}$ to the functions in $C_{2\pi}$ and he obtained some properties of the operators $L_{n,p}$ for $p = 3,4,5,6$. (We remark here that, the cases $p = 1$ and $p = 2$, correspond to the well known Féjer-and Jackson operators). Schurer and Steutel [57] studied certain constants associated with the approximation by these operators. Lorentz [36] used it to give a proof of Jackson theorems. Rathore [49] made a sup-norm study of certain linear combinations of $L_{n,p}$. Further, he [49] obtained an asymptotic formula for the error in the approximation of derivatives of functions by the corresponding derivatives of the operators.

In this chapter, we prove local direct and inverse theorems for $L_{n,p}(.,k,t)$ of orders $O(\varphi(n^{-1}))$ and $o(\varphi(n^{-1}))$ in L_q-norm $(1 \leq q < \infty)$.

A sectionwise summary of this chapter is as follows : Section 5.2 contains basic results about the operator $L_{n,p}$ and $L_{n,p}(.,k,t)$. Section 5.3 consists of local direct theorem for $L_{n,p}(.,k,t)$ and Sections 5.4 and 5.5, respectively, consist of local $O(\varphi)$ and $o(\varphi)$-inverse theorems for the operator $L_{n,p}(.,k,t)$.

5.2 Basic Results.

Throughout this chapter, we denote $I_j = [a_j, b_j]$, $j=1,2$ and

$$K_n(u) = \frac{1}{A_{n,p}} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} \text{, where } a_1 < a_2 < b_2 < b_1.$$

Lemma 5.2.1 [49] . Let $m \in \mathbb{N}$ and $2p > m+1$. Then

$$\int_{-\pi}^{\pi} |u|^m K_n(u) \, du \leq C_{m,p} \, n^{-m},$$

where $C_{m,p}$ is a constant independent of n.

Lemma 5.2.2 [49] . If $r \leq m < 2p-2$, then

$$(5.2.1) \qquad \int_{-\pi}^{\pi} u^r K_n(u) \, du = \sum_{k=r}^{m} \frac{\beta_k}{n^k} + \frac{\varepsilon_n}{n^m},$$

where β_k's are independent of n and $\varepsilon_n \to 0$ as $n \to \infty$.

We can easily observe that if r is odd then the left hand side of (5.2.1) is identically equal to zero.

We prove the convergence of the operators $L_n(\cdot, t)$ for functions in $L_{q,2\pi}$ in

Lemma 5.2.3. Let $f \in L_{q,2\pi}$. Then

$$||L_{n,p}(f; t) - f(t)||_{L_q[-\pi,\pi]} \to 0 \text{ as } n \to \infty.$$

Proof. Let $\varepsilon > 0$. Then, there exists $\delta > 0$ such that, whenever $|u| < \delta$,

$$||f(t+u) - f(t)||_{L_q[-\pi,\pi]}^q \leq \varepsilon.$$

Now,

$$Z = ||L_{n,p}(f;t) - f(t)||^q_{L_q[-\pi,\pi]}$$

$$= \int_{-\pi}^{\pi} | \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} (\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}})^{2p} \{f(t+u)-f(t)\}du|^q \, dt,$$

which by Jensen's inequality and Fubini's theorem

$$\leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}})^{2p} |f(t+u)-f(t)|^q \, dt \, du$$

$$\leq \frac{1}{A_{n,p}} \int_{|u|<\delta} \int_{-\pi}^{\pi} |f(t+u)-f(t)|^q (\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}})^{2p} \, dt \, du$$

$$+ \frac{1}{A_{n,p}} \int_{|u|>\delta} \int_{-\pi}^{\pi} |f(t+u)-f(t)|^q (\frac{\sin \frac{nu}{2}}{\sin \frac{n}{2}})^{2p} \, dt \, du$$

$$< \varepsilon/2 + \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \int_{|u|>\delta} (\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}})^{2p} \frac{|u|^m}{\delta^m} \times$$

$$\times |f(t+u) - f(u)|^q \, du \, dt,$$

for any m > 0.

Thus, from lemma 5.2.1

$$Z < \varepsilon/2 + \frac{M_2 \, ||f||^q}{\delta^m} n^{-m}, \text{ whenever } 2p > m+1.$$

Now, choosing n sufficiently large, we get that $Z < \varepsilon$, and hence the result.

The L_p-boundedness is given by the following

$$\leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i} \{\int_{c}^{d} |f(t+u)|^{q} \frac{|u|^{\ell q}}{\delta^{\ell q}} dt\}^{1/q} du,$$

noting that for $t \ \varepsilon \ (c,d)$ and $u \ \varepsilon \ (-\delta,\delta)$, $t+u \ \varepsilon \ (a,b)$. Thus

$$Z \leq \frac{1}{A_{n,p}} \frac{1}{\delta^{\ell}} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+\ell} \{\int_{c}^{d} |f(t+u)|^{q} dt\}^{1/q} du.$$

Now, using lemma 5.2.1 and the fact that f is periodic, for $2p > i+\ell+1$, we get

$$Z \leq \frac{M}{n^{i+\ell}} ||f||_{L_{q}[-\pi,\pi]}.$$

Now the result follows, since ℓ is arbitrary but $< 2p-1$.

We close this section by stating a theorem of Rathore [49] about linear combinations.

<u>Theorem 5.2.6.</u> Let f be bounded in the interval $[-\pi+t,\pi+t]$ possessing an r-th derivative at t, $r \leq k < 2p-2$. Then

$$|L_{n,p}(f,k,t) - f(t)| = o(n^{-r})$$

and $(n \to \infty)$.

$$|L_{n,p}(f,r-1,t) - f(t)| = O(n^{-r}).$$

Rathore proved the above theorem for the particular linear combinations taking $d_{j} = j+1$. But, same proof ramains valid for linear combinations defined in (5.1.3) as well.

5.3 Direct theorem.

We prove a direct theorem in this section which contains

$O(\varphi)$ and $o(\varphi)$ direct theorems as corollaries. The corresponding inverse theorems will be proved in Sections 5.4 and 5.5, respectively.

Theorem 5.3.1. Let $f \in L_{q,2\pi}$ and $p > \frac{k+2}{2}$. Then,

$$||L_{n,p}(f,k,t)-f(t)||_{L_q(I_2)}$$

$$\leq M\{\omega_{k+1}(f,n^{-1},q,I_1) + n^{-(k+1)}||f||_{L_q[-\pi,\pi]}\}.$$

Before proving the theorem, we prove the following results which shall be used in the proof of the theorem.

Lemma 5.3.2. Let $f \in L_{q,2\pi}$ and $i,j \in \mathbb{N}^O$ such that $i+j+2 < 2p$. Then, for any $\ell > 0$ such that $2p > \ell+1$,

$$||L_{n,p}(|u-t|^i|\int_t^u |u-w|^j |f(w)|dw|;t)||_{L_q(I_2)}$$

$$\leq M\{n^{-(i+j+1)}||f||_{L_q(I_1)} + n^{-\ell}||f||_{L_q[-\pi,\pi]}\}.$$

Proof. Suppose $q > 1$. Let X be the characteristic function of I_1. Then

$$J = ||L_{n,p}(|u-t|^i|\int_t^u |u-w|^j |f(w)|dw|;t)||_{L_q(I_2)}$$

$$\leq ||L_{n,p}(X(u) |u-t|^i |\int_t^u |u-w|^j f(w)|dw|;t)||_{L_q(I_2)}$$

$$+ ||L_{n,p}((1-X(u))|u-t|^i|\int_t^u |u-w|^j|f(w)|dw|;t)||_{L_q(I_2)}$$

$$(5.3.1) \qquad = J_1 + J_2, \text{ say.}$$

First we estimate J_2. Let $\delta = \min (b_1-b_2, a_2-a_1)$. Then clearly $|u-t| \geq \delta$ for all $t \varepsilon I_2$ and $\not\in I_1$. Hence

$$J_2 \leq \frac{1}{A_{n,p}} \{\int_{a_2}^{b_2} |\int_{-\pi}^{\pi} (\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}})^{2p} (1-X(t+u)) |u|^{i+j} \times$$

$$\times |\int_{t}^{t+u} f(w) |dw| du|^q dt\}^{1/q}$$

and hence by lemma 0.6.7 and Jensen's inequality, for $m > 0$

$$J_2 \leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} (\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}})^{2p} \frac{|u|^{i+j+m}}{\delta^m} \{\int_{a_2}^{b_2} |u|^{q-1} \times$$

$$\times |\int_{t}^{t+u} |f(w)|^q dw| dt\}^{1/q} du.$$

Now, using the periodicity of f, from lemma 5.2.1, we get

$$J_2 \leq \frac{M_1}{n^{i+j+m}} ||f||_{L_q[-\pi,\pi]} ,$$

whenever $i+j+m-1 < 2p$.

Choosing $\ell = i+j+m$, we get

$$(5.3.2) \qquad J_2 \leq \frac{M_2}{n^\ell} ||f||_{L_q[-\pi,\pi]} .$$

Now,

$$J_1 \leq ||L_{n,p}(X(u) |u-t|^{i+j} |\int_{t}^{u} |f(w) |dw| ;t) ||_{L_q(I_2)}$$

$$\leq ||L_{n,p}(X(u) |u-t|^{i+j+1} |H_{|f|}(t)| ;t) ||_{L_q(I_2)} ,$$

where $H_{|f|}$ is the Hardy-Littlewood majorant of the function f over the interval I_1.

Now, since $2p > i+j+2$, Jensen's inequality and lemma 5.2.1, imply that

$$(5.3.2) \qquad J_1 \leq \frac{M_3}{n^{i+j+1}} \, ||H_i f||_{L_q(I_1)}$$

$$\leq \frac{M_3}{n^{i+j+1}} \, ||f||_{L_q(I_1)} \, ,$$

by lemma 0.6.2.

Suppose $q = 1$. Then

$$J \leq \frac{1}{A_{n,p}} \int_{a_2}^{b_2} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j} \, |\int_{t}^{t+u} |f(w)| \, dw| \, du \, dt$$

$$= \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j} \int_{a_2}^{b_2} |\int_{t}^{t+u} |f(w)| \, dw| \, dt \, du$$

$$\leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j} |u| \int_{a_2-|u|}^{a_2+|u|} |f(w)| \, dw \, du$$

$$\leq \frac{1}{A_{n,p}} \int_{|u|<\delta} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j+1} \int_{a_2-|u|}^{b_2+|u|} |f(w)| \, dw \, du$$

$$+ \frac{1}{A_{n,p}} \int_{|u|\geq\delta} \left(\frac{\sin \frac{nu}{2}}{\sin \frac{u}{2}}\right)^{2p} |u|^{i+j+1} \int_{a_2-|u|}^{b_2+|u|} |f(w)| \, dw \, du,$$

where $\delta = \min(a_2-a_1, b_1-b_2)$.

Since $2p > i+j+2$ and $2p > \ell+1$, by lemma 5.2.1,

$$(5.3.3) \qquad J \leq \frac{M_4}{n^{i+j+1}} \, ||f||_{L_1(I_1)} + \frac{M_5}{n^{\ell}} \, ||f||_{L_1[-\pi,\pi]} \, ,$$

and hence the result follows from (5.3.1-3).

Lemma 5.3.3. Let $f \varepsilon L_{q,2\pi}$. If f has $k+1$ derivatives on I_1 with $f^{(k)} \varepsilon A.C.(I_1)$, $f^{(k+1)} \varepsilon L_q(I_1)$ and $p > \frac{k+2}{2}$, then

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_2)}$$

$$\leq \frac{M}{n^{k+1}} \{||f^{(k+1)}||_{L_q(I_1)} + ||f||_{L_q[-\pi,\pi]}\},$$

where M is a constant.

Proof. For $t \varepsilon I_2$ and $u \varepsilon I_1$,

$$f(u) = \sum_{i=0}^{k} \frac{(u-t)^i}{i!} f^{(i)}(t) + \frac{1}{(k+1)!} \int_t^u (u-w)^k f^{(k+1)}(w) dw.$$

Hence, if X is the characteristic function of I_1, we have

$$L_{n,p}(f,t) = L_{n,p}(Xf;t) + L_{n,p}((1-X)f;t)$$

$$= \sum_{i=0}^{k} \frac{f^{(i)}(t)}{i!} L_{n,p}(X(u)(u-t)^i;t)$$

$$+ \frac{1}{k!} L_{n,p}(X(u) \int_t^u (u-w)^k f^{(k+1)}(w) dw;t)$$

$$+ L_{n,p}((1-X)f;t).$$

Hence,

$$L_{n,p}(f,k,t) - f(t) = \sum_{i=1}^{k} \{\frac{f^{(i)}(t)}{i!} L_{n,p}(X(u)(u-t)^i,k,t)$$

$$+ \frac{1}{k!} L_{n,p}(X(u) \int_t^u (u-w)^k f^{(k+1)}(w) dw,k,t)$$

$$+ L_{n,p}((1-X)f,k,t)$$

$(5.3.4)$ $\qquad = J_1 + J_2 + J_3,$ say.

Clearly, from lemma 5.2.5 and the boundedness of $C(j,k)$, for any ℓ such that $0 < \ell < 2p-1$, we get

$$(5.3.5) \quad ||J_3||_{L_q(I_2)} \leq \frac{M_1}{n^\ell} ||f||_{L_q[-\pi,\pi]} .$$

Proceeding as in the case of J_1 in lemma 5.3.2 and using the boundedness of $C(j,k)$ again, we get

$$(5.3.6) \quad ||J_2||_{L_q(I_2)} \leq \frac{M_2}{n^{k+1}} ||f^{(k+1)}||_{L_q(I_1)} .$$

Now,

$$L_{n,p}(X(u) (u-t)^i, k, t)$$

$$= L_{n,p}((X(u)-1)(u-t)^i, k, t) + L_{n,p}((u-t)^i, k, t)$$

$$(5.3.7) = Z_1 + Z_2,$$ say.

Proceeding as in lemma 5.2.5, for any ℓ such that $0 < \ell < 2p-1$, we get

$$(5.3.8) \quad ||Z_1||_{L_q(I_2)} \leq \frac{M_3}{n^\ell} ||f||_{L_q[-\pi,\pi]} .$$

Also, by lemma 5.2.2 and the fact that

$$\sum_{j=0}^{k} C(j,k) d_j^{-m} = 0, \quad m = 1,2,\ldots,k,$$ we get

$$(5.3.9) \quad ||Z_2||_{L_q(I_2)} \leq \frac{M_4}{n^{k+1}} .$$

Hence, combining (5.3.7-9) (taking $\ell = k+1$),

/

$$||J_1||_{L_q(I_2)} \leq \frac{M_5}{n^{k+1}} \left(\sum_{i=1}^{k} ||f^{(i)}||_{L_q(I_2)} + ||f||_{L_q[-\pi,\pi]} \right)$$

and hence by lemma 0.6.3 , we get

$$(5.3.10) \quad ||J_1||_{L_q(I_2)} \leq \frac{M_6}{n^{k+1}} \{ ||f^{k+1}||_{L_q(I_2)} + ||f||_{L_q(I_2)} \}.$$

The lemma, now, follows from (5.3.4) and the estimates (5.3.5), (5.3.9-10) of J_1, J_2, J_3.

Proof of theorem 5.3.1. We have

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_2)} \leq ||L_{n,p}(f - f_{\eta,k+1}, k, t)||_{L_q(I_2)}$$

$$+ ||L_{n,p}(f_{\eta,k+1}, k, t) - f_{\eta,k+1}(t)||_{L_q(I_2)}$$

$$+ ||f_{\eta,k+1}(t) - f(t)||_{L_q(I_2)} .$$

$$(5.3.11) \quad = Z_1 + Z_2 + Z_3, \text{ say.}$$

Let X be the characteristic function of $[x,y]$ where $a_1 < x < a_2 < b_2 < y < b_1$, then,

$$Z_1 \leq ||L_{n,p}(X(f - f_{\eta,k+1}), k, t)||_{L_q(I_2)}$$

$$+ ||L_{n,p}((1-X)(f - f_{\eta,k+1}), k, t)||_{L_q(I_2)} .$$

Hence, from lemmas 5.2.4-5 and 0.6.5, we get

$$(5.3.12) \quad Z_1 \leq M_2 \{ \omega_{k+1}(f, \eta, q, I_1) + n^{-(k+1)} ||f||_{L_q[-\pi,\pi]} .$$

Also, from Lemmas 5.3.3 and 0.6.5, we get

(5.3.13) $\quad Z_2 \leq \dfrac{M_3}{n^{k+1}} \{\eta^{-(k+1)} \omega_{k+1}(f,\eta,q,I_1) + ||f||_{L_q}[-\pi,\pi]\}.$

Again, from lemma 0.6.5 , we get

(5.3.14) $\quad Z_3 \leq M_4 \, \omega_{k+1}(f,\eta,q,[a_1,b_1]).$

Hence, taking $\eta = n^{-1}$ and combining (5.3.11-14), we get the result.

Corollary 5.3.4. Let $f \, \varepsilon \, L_{q,2\pi}$, $p > \dfrac{k+2}{2}$ and $\varphi \, \varepsilon \, \Phi_{k+1}$. Then,

$$\omega_{k+1}(f,t,q,I_1) = O(\varphi(t)) \quad (t \to 0)$$

implies that

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_2)} = O(\varphi(\tfrac{1}{n})) \quad (n \to \infty).$$

Corollary 5.3.5. Let $f \, \varepsilon \, L_{q,2\pi}$, $p > \dfrac{k+2}{2}$ and $\varphi \, \varepsilon \, \Phi_{k+1}$. Then

$$\omega_{k+1}(f,t,q,I_1) = o(\varphi(t)) \quad (t \to 0)$$

implies that

$$||L_{n,p}(f,k,t) - f(t)||_{L_q(I_2)} = o(\varphi(\tfrac{1}{n})) \quad (n \to \infty).$$

5.4 O(φ)-inverse theorem.

In this section we prove the corresponding O(φ)-inverse theorem of corollary 5.3.4.

Theorem 5.4.1. Let $f \, \varepsilon \, L_{q,2\pi}$, $\varphi \, \varepsilon \, \Phi_{k+1}$, and $p > \dfrac{2k+1}{2}$. Then

(5.4.1) $\quad ||L_{n,p}(f,k,t) - f(t)||_{L_q(I_1)} = O(\varphi(n^{-1})) \quad (n \to \infty)$

implies that

(5.4.2) $\omega_{k+1}(f,t,q,I_2) = O(\varphi(t))$ $(t \to O)$.

   Before going into the details of the proof, we prove the following

Lemma 5.4.2.   Let $f \in L_{q,2\pi}$. Then, for $i \in \mathbb{N}^o$ and $\ell > O$ such that $2p > i+1$, $\ell+1$,

$$||L_{n,p}(|u-t|^i|f(u)|;t)||_{L_q(I_2)}$$

$$\leq M\{n^{-i} ||f||_{L_q(I_1)} + n^{-\ell} ||f||_{L_q[-\pi,\pi]}\}.$$

Proof.   By generalised Minkowski inequality,

$$J = ||L_{n,p}(|u-t|^i|f(u)|;t)||_{L_q(I_2)}$$

$$\leq \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} \{\int_{a_2}^{b_2} (\frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}})^{2pq}\cdot|u|^{iq}|f(t+u)|^q dt\}^{\frac{1}{q}} du$$

$$= \frac{1}{A_{n,p}} \int_{-\pi}^{\pi} (\frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}})^{2p} |u|^i \{\int_{a_2}^{b_2} |f(t+u)|^q dt\}^{\frac{1}{q}} du .$$

Hence, if $\delta = \min (a_2-a_1, b_1-b_2)$ then

$$J_1 \leq \frac{1}{A_{n,p}} \int_{-\delta}^{\delta} (\frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}})^{2p} \{\int_{a_2}^{b_2} |f(t+u)|^q dt\}^{\frac{1}{q}} du$$

$$+ \frac{1}{A_{n,p}} \int_{\pi\geq|u|>\delta} (\frac{\sin\frac{nu}{2}}{\sin\frac{n}{2}})^{2p} |u|^i (\int_{a_2}^{b_2} |f(t+u)|^q dt)^{\frac{1}{q}} du$$

$$\leq \frac{M_1}{n^i} ||f||_{L_q(I_1)} + \frac{M_2}{n^\ell} ||f||_{L_q[-\pi,\pi]} ,$$

since f is periodic and $2p > \ell+1, i+1$. Hence the result.

Lemma 5.4.3. Let $f \in L_{q,2\pi}$ such that supp $f \subset I_2$. Then

$$||L_{n,p}^{(k+1)}(f;t)||_{L_q(I_2)} \leq M_1 \, n^{(k+1)} \, ||f||_{L_q(I_2)}.$$

Further, if f has $k+1$ derivatives with $f^{(k)} \in A.C.(I_2)$ and $f^{(k+1)} \in L_q(I_2)$ then

$$||L_{n,p}^{(k+1)}(f;t)||_{L_q(I_2)} \leq M_2||f^{(k+1)}||_{L_q(I_2)}.$$

Proof. We know that $L_{n,p}(f;t)$ is a trigonometric polynomial of degree $np-p$ for any $f \in L_{q,2\pi}$

Hence, by Bernstein inequality [65], since supp $f \subset I_2$, it follows that

$$||L_{n,p}^{(k+1)}(f;t)||_{L_q(I_2)} \leq (pn)^{k+1}||L_{n,p}(f;t)||_{L_q[-\pi,\pi]}$$

and hence the first part follows from lemma 5.2.4.

Since $f^{(k)} \in A.C.(I_2)$ and $f^{(k+1)} \in L_q(I_2)$

$$L_{n,p}^{(k+1)}(f;t) = L_{n,p}(f^{(k+1)};t),$$

and hence the second part of the result follows similarly.

Proof of theorem 5.4.1. Let x_i, y_i, $i = 1,2,3,4$ satisfy $a_1 < x_1 < x_2 < x_3 < x_4 < a_2 < b_2 < y_4 < y_3 < y_2 < y_1 < b_1$. We choose a function $g \in C_o^{k+1}$ such that supp $g \subset (x_3,y_3)$ and $g(t) = 1$ on $[x_4,y_4]$. Writing $fg = \bar{f}$, we have

$(5.4.3)$ $\quad ||\Delta_\gamma^{k+1} \bar{f}(t)||_{L_q[x_3,y_3]}$

$$\leq ||\Delta_\gamma^{k+1} \{\bar{f}(t) - L_{n,p}(\bar{f},k,t)\}||_{L_q[x_3,y_3]}$$

$$+ ||\Delta_\gamma^{k+1} L_{n,p}(\bar{f},k,t)||_{L_q[x_3,y_3]}.$$

By lemma $0.6.1$

$||\Delta_\gamma^{k+1} L_{n,p}(\bar{f},k,t)||_{L_q[x_3,y_3]}$

$$= ||\int_0^\gamma \cdots \int_0^\gamma L_{n,p}^{(k+1)}(\bar{f},k,t+\sum_{i=1}^{k+1} z_i) \, dz_1 \cdots dz_{k+1}||_{L_q[x_3,y_3]}.$$

By repeated applications of Jensen's inequality and Fubini's theorem, for all sufficiently small γ, we have

$$\int_{x_3}^{y_3} |\int_0^\gamma \cdots \int_0^\gamma L_{n,p}^{(k+1)}(\bar{f},k,t+\sum_{i=1}^{k+1} z_i) \, dz_1 \cdots dz_{k+1}|^q \, dt$$

$$\leq \gamma^{(k+1)(q-1)} \int_0^\gamma \cdots \int_0^\gamma |L_{n,p}^{(k+1)}(\bar{f},k,t+\sum_{i=1}^{k+1} z_i)|^q \, dt \, dz_1 \cdots dz_{k+1}$$

$$\leq \gamma^{(k+1)q} ||L_{n,p}^{(k+1)}(\bar{f},k,t)||_{L_q[x_2,y_2]}^q,$$

since γ is sufficiently small.

Hence,

$||\Delta_\gamma^{k+1} L_{n,p}(\bar{f},k,t)||_{L_q[x_3,y_3]}$

$$\leq \gamma^{k+1} ||L_{n,p}^{(k+1)}(\bar{f},k,t)||_{L_q[x_2,y_2]}$$

$$\leq \gamma^{k+1} \{||L_{n,p}^{(k=1)}(\bar{f}-\bar{f}_{\eta,k+1},k,t)||_{L_q[x_2,y_2]}$$

Hence by lemma 5.4.3, for all sufficiently small $\eta > 0$, we have

$$||\Delta_\gamma^{k+1} L_{n,p}(\vec{f},k,t)||_{L_q[x_3,Y_3]}$$

$$\leq M_1 \gamma^{k+1} \{n^{k+1} ||\vec{f}-\vec{f}_{\eta,k+1}||_{L_q[x_2,Y_2]}$$

$$+ ||\vec{f}_{\eta,k+1}^{-(k+1)}||_{L_q[x_2,Y_2]}\},$$

which, by lemma 0.6.5, implies that (using the fact that supp $\vec{f} \subset (x_3,Y_3)$)

$$(5.4.4) \quad ||\Delta_\gamma^{k+1} L_{n,p}(\vec{f},k,t)||_{L_q[x_3,Y_3]}$$

$$\leq M_2 \gamma^{k+1}(n^{k+1} + \frac{1}{\eta^{k+1}}) \omega_{k+1}(\vec{f},\eta,q,[x_3,Y_3]) .$$

Next we are to show that

$$(5.4.5) \quad ||\Delta_\gamma^{k+1} \{\vec{f}(t)-L_{n,p}(\vec{f},k,t)\}||_{L_q[x_3,Y_3]} = O(\varphi(n^{-1})) \quad (n \to \infty) .$$

After having proved this, we may combine with (5.4.3-4) to get

$$||\Delta_\gamma^{k+1} \vec{f}(t)||_{L_q[x_3,Y_3]}$$

$$\leq M_3 \{\varphi(n^{-1}) + \gamma^{k+1} (n^{k+1} + \frac{1}{\eta^{k+1}}) \omega_{k+1}(\vec{f},\eta,q [x_3,Y_3])\} .$$

Then choosing n such that $n \leq \eta^{-1} < 2n$ and taking supremum over all $\gamma \leq t$, we have

$$\omega_{k+1}(\vec{f},t,q, [x_3,Y_3])$$

$$\leq M_4\{\varphi(\eta) + (\frac{t}{\eta})^{k+1} \omega_{k+1}(\vec{f},\eta,q,[x_3,Y_3])\};$$

which, by lemma 1.2.2, proves the result.

Now, we prove (5.4.5) by induction as follows :

First, we prove the result for all $\varphi \varepsilon \Phi_1$. Later, assuming the validity of the result for all $\varphi \varepsilon \Phi_r$ for some r such that $1 \leq r \leq k$, we prove it for $\varphi \varepsilon \Phi_{r+1}$.

Let $\varphi \varepsilon \Phi_1$. Then

$$||L_{n,p}(fg,k,t) - (fg)(t)||_{L_q[x_3,y_3]}$$

$$\leq ||L_{n,p}((f(u)-f(t)) \, g(t),k,t)||_{L_q[x_3,y_3]}$$

$$+ ||L_{n,p}(f(u)(g(u)-g(t)),k,t)||_{L_q[x_3,y_3]}$$

$$= ||g(t)\{L_{n,p}(f,k,t) - f(t)\}||_{L_q[x_3,y_3]}$$

$$+ ||L_{n,p}(f(u)(u-t) \, g'(\xi),k,t)||_{L_q[x_3,y_3]} \cdot$$

for some ξ lying between u and t. Hence (5.4.1) and lemma 5.4.2 together imply that

$$||L_{n,p}(fg,k,t) - (fg)(t)||_{L_q[x_3,y_3]} \leq M_5 \{\varphi(\tfrac{1}{n}) + \tfrac{1}{n}\} \leq M_6 \, \varphi(\tfrac{1}{n}),$$

since $t/\varphi(t) \to 0$ as $t \to 0$.

Thus, we have proved the theorem for all $\varphi \varepsilon \Phi_1$.

Now, assume the result for all $\varphi \varepsilon \Phi_r$ for some r ε IN such that $1 \leq r \leq k$ and let $\varphi \varepsilon \Phi_{r+1}$.

Now,

$$Z = ||L_{n,p}(fg,k,t) - (fg)'(t)||_{L_q[x_3,y_3]}$$

$$\leq ||L_{n,p}(f(u)(g(u)-g(t)),k,t)||_{L_q[x_3,y_3]}$$

$$+ ||L_{n,p}((f(u)-f_{\eta,k+1}(u))(g(u)-g(t)),k,t)||_{L_q[x_3,y_3]}$$

$$+ ||L_{n,p}((f_{\eta,k+1}(u)-f_{\eta,k+1}(t))(g(u)-g(t),k,t)||_{L_q[x_3,y_3]}$$

$$+ ||L_{n,p}(f_{\eta,k+1}(g(u)-g(t)),k,t)||_{L_q[x_3,y_3]}$$

$$(5.4.6) \quad = Z_0 + Z_1 + Z_2 + Z_3, \text{ say.}$$

Clearly,

$$(5.4.7) \quad Z_0 \leq M_7 \, \varphi(\tfrac{1}{n}).$$

By theorem 5.2.6 and lemma 0.6.5

$$(5.4.8) \quad Z_3 \leq \frac{M_8}{k+1} \, ||f||_{L_q[-\pi,\pi]}.$$

Moreover, for some ξ lying between u and t,

$$Z_1 = ||L_{n,p}(f(u)-f_{\eta,k+1}(u))(u-t)\,g'(\xi),k,t)||_{L_q[x_3,y_3]}, \text{ from}$$

which by lemma 5.4.2 for any ℓ such that $0 < \ell < 2p-1$, we get

$$Z_1 \leq M_9 \{n^{-1} \, ||f-f_{\eta,k+1}||_{L_q[x_2,y_2]} + n^{-\ell} \, ||f||_{L_q[-\pi,\pi]}\}.$$

Now, applying lemma 0.6.5.

$$(5.4.9) \quad Z_1 \leq M_{10} \{ n^{-1} \omega_{k+1}(f, \eta, q[x_1, y_1] + n^{-\ell} \|f\|_{L_q}[-\pi, \pi] \}.$$

After Taylor's series expansion, we observe that

$$Z_2 \leq \frac{1}{k!} \quad \|L_{n,p}((\sum_{i=1}^{k-1} \frac{g^{(i)}(t)}{i!} (u-t)^i) \times$$

$$\times (\int_t^u (u-w)^k f_{\eta,k+1}^{(k+1)}(w)\,dw), k, t)\|_{L_q}[x_3, y_3]$$

$$+ \frac{1}{(k!)^2} \|L_{n,p}(g^{(k)}(\xi)(u-t)^k \times$$

$$\times \int_t^u (u-w)^k f_{\eta,k+1}^{(k+1)}(w)\,dw, k, t)\|_{L_q}[x_3, y_3]$$

$$+ \sum_{i=1}^{k} \sum_{j=1}^{k-1} \frac{1}{i!\,j!} \|f_{\eta,k+1}^{(i)}(t) g^{(j)}(t) L_{n,p}((u-t)^{i+j}, k, t)\|_{L_q}[x_3, y_3]$$

$$+ \frac{1}{k!} \{ \sum_{i=1}^{k} \frac{1}{i!} \|f_{\eta,k+1}^{(i)}(t) L_{n,p}((u-t)^{k+i} g^{(k)}(\xi), k, t)\|_{L_q}[x_3, y_3]$$

$$(5.4.10) \quad = R_1 + R_2 + R_3 + R_4, \text{ say.}$$

By lemma 5.3.2, for any ℓ such that $0 < \ell < 2p-1$, we get

$$(5.4.11) \quad R_1 \leq M_{10} \{ \sum_{i=1}^{k-1} n^{-(k+i)} \|f_{\eta,k+1}^{(k+1)}\|_{L_q}[x_1, y_1]$$

$$+ n^{-\ell} \|f_{\eta,k+1}^{(k+1)}\|_{L_q}[-\pi, \pi] \}$$

and

$$(5.4.12) \quad R_3 \leq M_{11} \{ n^{-(2k+1)} \|f_{\eta,k+1}^{(k+1)}\|_{L_q}[x_1, y_1]$$

$$+ n^{-\ell} \|f_{\eta,k+1}^{(k+1)}\|_{L_q}[-\pi, \pi] \}.$$

It follows from lemma 5.2.2 and the fact that $\sum_{j=0}^{k} C(j,k)\, d_j^{-m} = 0$, $m = 1,2,\ldots,k$, that

$$R_3 \leq \frac{M_{12}}{n^{k+1}} \left(\sum_{i=1}^{k} ||f_{\eta,k+1}^{(i)}||_{L_q[x_3,y_3]} \right).$$

Hence by lemma 0.6.3

(5.4.13) $\quad R_3 \leq \frac{M_{13}}{n^{k+1}} \left\{ ||f_{\eta,k+1}^{(k)}||_{L_q[x_3,y_3]} + ||f_{\eta,k+1}||_{L_q[x_3,y_3]} \right\}$

and

(5.4.14) $\quad R_4 \leq \frac{M_{14}}{n^{k+1}} \left\{ ||f_{\eta,k+1}^{(k)}||_{L_q[x_3,y_3]} + ||f_{\eta,k+1}||_{L_q[x_3,y_3]} \right\}.$

Choosing n such that $n < \eta^{-1} \leq 2n$, it follows from (5.4.11–14) and lemma 0.6.5 , that

(5.4.15) $\quad R_1, R_2 \leq M_{16}\{n^{-1} \omega_k(f,n^{-1},q,[x_1,y_1]) + n^{-(k+1)} ||f||_{L_q[-\pi,\pi]}\}$

and

(5.4.16) $\quad R_3, R_4 \leq M_{17}\{n^{-1} \omega_{k+1}(f,n^{-1},q[x_1,y_1]) + n^{-(k+1)} ||f||_{L_q[-\pi,\pi]}\}.$

Now, clearly $\frac{\varphi(t)}{t} \varepsilon \; \Phi_r$ and hence by induction hypothesis, we have

$$\omega_{k+1}(f,t,q,[x_1,y_1]) = O(\frac{\varphi(t)}{t}) \qquad (t \to 0)$$

and hence, by lemma 1.2.4, we get

$$\omega_k(f,t,q,[x_1,y_1]) = O(\frac{\varphi(t)}{t}) \qquad (t \to 0).$$

Thus, for $j = k, k+1,$

(5.4.17) $n^{-1} \omega_j(f, n^{-1}, q, [x_1, y_1]) = O(\varphi(n^{-1}))$ $(n \to \infty)$.

Also since $\varphi \in \Phi_{r+1} \subseteq \Phi_{k+1}$,

(5.4.18) $\dfrac{n^{-(k+1)}}{\varphi(n^{-1})} \to 0$ as $n \to \infty$.

Hence, in view of (5.4.17-18), combining (5.4.7-10) and (5.4.15-16), we get

$$Z = Z_0 + Z_1 + Z_2 + Z_3 \le M_{19} \, \varphi(n^{-1})$$

and hence the theorem is proved.

Corollary 5.4.4. Let $f \in L_{q, 2\pi}$ and $1 \le r \le k+1$ and let $\varphi \in \Phi_r$. Then

$$\|L_{n,p}(f, k, t) - f(t)\|_{L_q(I_1)} = O(\varphi(n^{-1}))$$

implies that

$$\omega_r(f, t, q, I_1) = O(\varphi(t)) \quad (t \to 0).$$

5.5 $o(\varphi)$-inverse theorem.

In section 5.3 we proved a $o(\varphi)$-direct theorem (corollary 5.3.4). In this section, we prove the corresponding $o(\varphi)$-inverse theorem.

Theorem 5.5.1. Let $f \in L_{q, 2\pi}$, $p > \dfrac{2k+1}{2}$ and $\varphi \in \Phi_{k+1}$. Then

(5.5.1) $\|L_{n,p}(f, k, t) - f(t)\|_{L_q(I_1)} = o(\varphi(n^{-1}))$ $(n \to \infty)$

implies that

(5.5.2) $\omega_{k+1}(f, t, q, I_2) = o(\varphi(t))$ $(t \to 0)$.

<u>Proof.</u>   Choosing $x_i, y_i, g, \bar{f}$ as in the proof of theorem 5.4.1 and proceeding as in the proof of the same theorem, we get

$$(5.5.3) \quad ||\Delta_\gamma^{k+1} L_{n,p}(\bar{f}, k, t)||_{L_q[x_3, y_3]}$$

$$\leq M_1 \gamma^{k+1} (n^{k+1} + \frac{1}{\eta^{k+1}}) \omega_{k+1}(\bar{f}, \eta, q, [x_3, y_3]).$$

Next we have to show that

$$(5.5.4) \quad ||\Delta_\gamma^{k+1} \{\bar{f}(t) - L_{n,p}(\bar{f}, k, t)\}||_{L_q[x_3, y_3]} = o(\varphi(n^{-1}))(n \to \infty).$$

Now define $\psi(x)$ as follows :

$$||\Delta_\gamma^{k+1}\{\bar{f}(t) - L_{n,p}(\bar{f}, k, t)||_{L_q[x_3, y_3]} \text{ if } x = n^{-1}$$

$$(5.5.5) \quad \psi(x) = \{$$

$$\psi(n^{-1}) \qquad\qquad \text{ if } x \in ((n+1)^{-1}, n^{-1}).$$

Clearly $\psi(x) = o(\varphi(x)) \ (x \to 0)$ and hence, after having proved (5.5.4) combining (5.5.3-5), we get

$$||\Delta_\gamma^{k+1} \bar{f}(t)||_{L_q[x_3, y_3]}$$

$$\leq M_2 \{\psi(n^{-1}) + \gamma^{k+1}(n^{k+1} + \frac{1}{\eta^{k+1}}) \omega_{k+1}(\bar{f}, \eta, q, [x_3, y_3])\}.$$

Now, choosing n such that $n \leq \eta^{-1} < n+1$ and taking supremum over all $\gamma \leq t$, we have

$$\omega_{k+1}(\bar{f}, t, q, [x_3, y_3])$$

$$\leq M_4 \{\psi(\eta) + (\frac{t}{\eta})^{k+1} \omega_{k+1}(f, \eta, q, [x_3, y_3])\}.$$

Now, conclusion (5.5.2) follows from lemma **1.2.3.**

Thus, we are left to show that

(5.5.6) $\quad ||fg(t) - L_{n,p}(fg,k,t)||_{L_q[x_3,y_3]} = o(\varphi(n^{-1}))\quad (n \to \infty).$

As in the proof of theorem 5.4.1, we prove this by
induction as follows : First, we prove the theorem for all
$\varphi \,\varepsilon\, \Phi_1$. Next, assuming the result of the theorem for all
$\varphi \,\varepsilon\, \Phi_r$ for some r such that $1 \le r \le k$, we prove the result for
all $\varphi \,\varepsilon\, \Phi_{r+1}$.

Let $\varphi \,\varepsilon\, \Phi_1$. Then, from hypothesis and lemma 5.4.2, we
have

$||(fg)(t) - L_{n,p}(fg,k,t)||$

$\qquad \le ||L_{n,p}((f(u)-f(t))\,g(t),k,t)||_{L_q[x_3,y_3]}$

$\qquad + ||L_{n,p}(f(u)(g(u)-g(t)),k,t)||_{L_q[x_3,y_3]}$

$\qquad \le M_3 \{\varphi(\frac{1}{n}) + \frac{1}{n}\}$

and hence (5.5.6) is proved, for all $\varphi \,\varepsilon\, \Phi_1$. Thus, the
theorem is proved for all $\varphi \,\varepsilon\, \Phi_1$.

Now, assume that the theorem holds for all $\varphi \,\varepsilon\, \Phi_r$ for
some r such that $1 \le r \le k$ and let $\varphi \,\varepsilon\, \Phi_{r+1}$.
Then, clearly

$$\frac{\varphi(t)}{t} \,\varepsilon\, \Phi_r.$$

Hence, by induction hypothesis, we have

$$\omega_{k+1}(f,t,q,[x_1,y_1]) = o(\frac{\varphi(t)}{t}) \quad (t \to 0),$$

which, by lemma 1.2.5, implies that

$$\omega_k(f,t,q,[x_1,y_1]) = o(\frac{\varphi(t)}{t}) \quad (t \to 0).$$

Thus, we get, for $j = k$, $k+1$,

(5.7) $\qquad n^{-1} \omega_j(f, \frac{1}{n}, q,[x_1,y_1]) = o(\varphi(\frac{1}{n})) \quad (n \to \infty).$

Also, as $\varphi \in \Phi_{k+1}$, we have

(5.8) $\qquad n^{-(k+1)}/\varphi(n^{-1}) \to 0$ as $n \to \infty$.

Now, proceeding again as in the proof of theorem 5.4.1, we get

$$L_{n,p}(fg,k,t)-(fg)(t)||_{L_q[x_3,y_3]} \leq Z_0 + Z_1 + Z_2 + Z_3 .$$

Clearly, by hypothesis (5.5.1)

$$Z_0 = o(\varphi(\frac{1}{n})) \quad (n \to \infty).$$

Also, from theorem 5.2.6 and lemma 0.6.5 and (5.5.8), we get

$$Z_3 = o(\varphi(\frac{1}{n})) \quad (n \to \infty).$$

Z_1, Z_2 were estimated in the proof of theorem 5.4.1 as

$$Z_1 \leq M_5\{n^{-1} \omega_{k+1}(f,n,q,[x_1,y_1]) + n^{-\ell} ||f||_{L_q}[-\pi,\pi]\}$$

$$2 \leq R_1 + R_2 + R_3 + R_4, \text{ where}$$
$$M_6\{n^{-1} \omega_k(f,n^{-1},q,[x_1,y_1]) + n^{-(k+1)} ||f||_{L_q}[-\pi,\pi]\}$$

and

$$R_3, R_4 \leq M_8 \{ n^{-1} \omega_{k+1}(f, n^{-1}, q, [x_1, y_1]) + n^{-(k+1)} ||f||_{L_q}[-\pi, \pi] \}.$$

Thus, from (5.5.7) we see that

$$Z_1, Z_2 = o(\varphi(\tfrac{1}{n}))$$

and hence the result.

We conclude this chapter with the following two corollaries :

<u>Corollary</u> 5.5.2. Let $f \in L_{q, 2\pi}$, $p > \frac{2k+1}{2}$ and $\varphi \in \Phi_r$ for some r such that $1 \leq r \leq k+1$. Then

$$||L_{n,p}(f, k, t) - f(t)||_{L_q(I_1)} = o(\varphi(\tfrac{1}{n}))$$

implies that

$$\omega_{k+1}(f, t, q, I_2) = o(\varphi(t)) \quad (t \to 0).$$

<u>Corollary</u> 5.5.3 (Global φ-inverse theorems). Let $f \in L_{q, 2\pi}$ and $p > \frac{2k+1}{2}$. Let $\varphi \in \Phi_{k+1}$. Then

$$||L_{n,p}(f, k, t) - f(t)||_{L_q}[-\pi, \pi] = O(\varphi(\tfrac{1}{n})) \quad (n \to \infty)$$

if and only if

$$\omega_{k+1}(f, t, q, [-\pi, \pi]) = O(\varphi(t)) \quad (t \to 0).$$

Moreover, above statement remains valid if O is replaced by o everywhere.

<u>Proof</u>.   Considering an interval $[a_1, b_1]$ where $a_1 < -\pi < \pi < b_1$ and using the periodicity of f, we get the result from the earlier $O(\varphi)$ and $o(\varphi)$-inverse theorems (Theorems 5.4.1 and 5.5.1).

REFERENCES

1. Agrawal P.N. :   Direct, Inverse and Saturation Theorems
                      in ordinary and Simultaneous Approximation
                      by Iterative Combinations of exponential
                      type operators, Ph.D. Thesis, I.I.T.
                      Kanpur (1979).

2. Becker M. :   Global approximation theorems for
                      Szas'z-Mirakjan and Baskakov operators
                      in polynomial weight spaces, Indiana
                      Univ. Math. J, 27(1978), 127-142.

3. Becker, M. :   An elementary proof of the inverse
                      theorem for Bernstein polynomials,
                      Aequationes Mathematicae, 19(1979),
                      145-150.

4. Becker, M., Kucharski, D. and Nessel, R.J. :
                      Global approximation theorem for the
                      Szas'z-Mirakjan operators in exponential
                      weight spaces, in: ''Linear spaces and
                      Approximation'' ISNM, Vol. 40, pp. 319-333,
                      Birkhàuser, Basel, 1978.

5. Becker, M. and Nessel, R.J. :
                      A global approximation theorem for Meyer-
                      Konig and Zeller operators, Math. Z.,
                      160(1978), 195-206.

6. Becker, M. and Nessel, R.J. :
                      An elementary approach to inverse
                      approximation theorems, J. Approx. Theory,
                      23 (1978), 99-103.

7. Becker, M. and Nessel, R.J. :
                      On global saturation for Kantorovitch
                      polynomials, in: Approximation and
                      ''Function spaces''(Proc. Conference,
                      Gdansk, 1979).

8. Becker, M. and Nessel, R.J. :
                      On global approximations by Kantorovitch
                      polynomials, in: Approximation Theory
                      III'', Academic Press, New York (1980).

9. Berens, H. and Devore, R:
                      Quantitative Korovkin theorems for
                      L_p-spaces in: ''Approximation Theory II'',
                      Academic Press, New York (1976).

10. Berens, H. and Lorentz, G.G.:
        Inverse theorems for Bernstein polynomials,
        Indiana Univ. Math. J, 21(1972), 693-708.

11. Bergh, J. and Löfström, J.:
        Interpolation spaces, Springer-Verlag,
        New York (1976).

12. Bleimann, G., Junggeburth, J. and Stark, E.L. :
        Improved approximation and Jackson's
        theorem , Technical report, Aachen (1979).

13. Bojanic, R. and Shisha, O. :
        Degree of L_1 approximation to integrable
        functions by modified Bernstein polynomials,
        J. Approx. Theory, 13(1975), 66-72.

14. Butzer, P.L. : Linear combinations of Bernstein polynomials,
        Can. J. Math. 5(1953), 559-567.

15. Butzer, P.L. : Dominated convergence of Kantorovitch
        polynomials in L_p, Trans. Roy. Soc. Canada,
        46 (1952), 23-27.

16. Butzer, P.L. and Berens, H. :
        Semi-Groups of operators and Approximation,
        Springer-Verlag, New York (1967).

17. Butzer, P.L. and Nessel, R.J. :
        Fourier Analysis and Approximation, Vol. I,
        Academic Press, New York (1971).

18. de Leeuw, K. : On the degree of approximation by
        Bernstein polynomials, J. d'Analyse
        Math., 7(1959), 89-104.

19. De Vore, R.A. : The Approximation of Continuous Functions
        by Positive Linear-Operators, Lecture
        Notes (293), Springer-Verlag, Berlin-
        New York (1972).

20. Ditzian, Z. : Inverse theorems for functions in L_p and
        other spaces, Proc. Amer. Math. Soc.,
        54(1976), 80-82.

21. Ditzian, Z. : A global inverse theorem for combinations
        of Bernstein polynomials, J. Approx.
        Theory, 26(1979), 277-292.

22. Ditzian, Z. and May, C.P. :
L_p-saturation and inverse theorems for modified Bernstein polynomials, Indiana Univ. Math. J., 25(1976), 733-751.

23. Ditzian, Z. and Freud, G. :
Linear approximating Processes with limited oscillation, J. Approx. Theory, 12(1974), 23-31.

24. Grundmann, A.:
Gutoabschatzungen fur den Kantorovic operator in der L_1-norm, Math. Z. 150(1976), 45-47.

25. Hildebrand, F.B.: Introduction to Numerical Analysis, Mc-Graw-Hill, New York (1956).

26. Hoeffding, W.: The L_1-norm of the approximation error for Bernstein-type polynomials, J. Approx. Theory, 4(1971), 347-356.

27. Hoff, J.C. : Approximation with kernels of finite oscillations (i) convergence, J. Approx. Theory, 3(1970) 213-218.

28. Hoff, J.C. : Approximation with kernels of finite oscillations (ii) Degree of approximation, J. Approx. Theory, 12(1974), 127-145.

29. Kantorovitch,L.V.: Sur certaines developpements suivant les polynomes de la forme de S. Bernstein, I,II, C.R. Acad. Sci, URSS. A (1930), 563-568 and 595-600.

30. Korovkin,P.P.: Convergence of linear positive operators in the space of continuous functions, D.A.N. 90(1953), 961-964 (Russian).

31. Korovkin : Linear operators and Approximation Theory, Hindustan Publishing Corporation, Delhi (1960).

32. Kunwar, B. : A class of Linear Positive Approximation Methods, Ph.D. Thesis, I.I.T. Kanpur (1979).

33. Kunwar, B. : On a class of linear positive operators, Indian Journal of Pure and applied Mathematics, 13(1): 13-24 (1982).

34. Lorentz, G.G. : Zur Theorie der polynome Von S. Bernstein, Mat. Sb., 2(1937), 543-556.

35. Lorentz, G.G. : Bernstein Polynomials, Univ. of Toronto Press, Toronto (1953).

36. Lorentz, G.G. : Approximation of Functions, Holf, Rinehart and Winston, New York (1966).

37. Maier, V. : The L_1 Saturation class of the Kanotorovic operators, J. Approx. Theory, 22 (1978), 223-232.

38. Maier, V. : L_p-approximation by Kantorovic operators, Math. Anal, 4(1978), 289-295.

39. May, C.P. : Saturation and Inverse theorems for combinations of a class of exponential type operators, Can. J. Maths., Vol. XXVIII, No. 6 (1976), 1224-1250.

40. May, C.P. : A saturation theorem for modified Bernstein polynomials in L_p-spaces, SIAM J. Math. Anal., Vol. 10, No. 2(1979), 321-330.

41. Micchelli, C.A.: Saturation classes and iterates of operators, Dessertation, Stanford University, 1969.

42. Micchelli, C.A.: The der Gamma iterates of Bernstein Polynomials, J. Approx. Theory, 8(1973), 1-18.

43. Muller, M.W. : Die Folge der Gamma operatoren, Thesis, Technische Hochschule, Stuttgart (1967).

44. Müller, M.W. : Die Gute der L_p-Approximation durch Kantorovic polynome, Math. Z., 152(1976), 243-247.

45. Muller, M.W. : L_p-approximation by the method of integral Meyer-König and Zeller operators, Studia Mathematica T. LXIII (1978), 81-88.

46. Müller, M.W. and Maier, V. :
Die lokale L_p-saturation klasse des verfahrens der integralen Meyer-Konig und Zeller operatoren, in: ''Linear Spaces and Approximation''(Editors P.L. Butzer and B.Sz-Nagy) ISNM 40, Birkhauser Verlag, Basel (1978).

47. Natanson, I.P.: Constructive Function Theory, Vol. I, Frederick Unger Publ. Co., New York (1964).

48. Rathore, R.K.S.: Linear combinations of linear positive operators and Generating Relations in Special Functions, Ph.D. Thesis, I.I.T. Delhi (1973).

49. Rathore, R.K.S.: On a sequence of Linear Trigonometric polynomial operators, SIAM J. Math. Anal. Vol. 5, No. 6 (1974), 908-917.

50. Rathore, R.K.S.: Approximation of Unbounded Functions with Linear Positive Operators, Doctoral Thesis, Technische Hogeschool, Delft, (1974).

51. Rathore, R.K.S.: (L,p)-Summability of a Multiply Differentiated Fourier Series, Indag, Math. 38(3) (1976) 217-250.

52. Rathore, R.K.S. and Agrawal, P.N.: Inverse and saturation theorems for derivatives of exponential type operators, Indian Journal of pure and Applied Mathematics, 13(4) (1982), 476-490.

53. Rathore, R.K.S. and Singh, O.P. : On convergence of derivatives of Post-Widder Operators, Indian Journal of Pure and Applied Mathematics, 11(5), (1980), 547-561.

54. Riemenschneider, S.D.: The L_p-saturation of Kantorovic-Bernstein polynomials, J. Approx. Theory, 23(1975), 158-162.

55. Schurer, F. : Some remarks on the approximation of functions by some positive linear operators, Monat. fur Math. Bd 67/4, 1963, 355-358.

56. Schurer, F. :   On Linear Positive Operators in
                      Approximation Theory, Doctoral Thesis,
                      Technische Hogeschool Delft, 1963.

57. Schurer, F. and Steutel, F.W. :
                      On Linear Positive operators of the
                      Jackson type, Mathematical communications,
                      Technological University of Twente,
                      no. 1, October 1966,

58. Sikkema, P.C. and Rathore, R.K.S. :
                      Convolutions with Powers of Bell-shaped
                      Functions, Report, Deptt. of Maths.,
                      Technische Hogeschool Delft, March 1976.

59. Sikkema, P.C. :   Fast Approximation by means of
                      convolution operators, Indag. Math,
                      43(1981) 431-444.

60. Singh, O.P. :   Some Approximation Theoretic Aspects
                      of the Post-Widder Inversion of Laplace
                      Transform, Ph.D. Theis, I.I.T. Kanpur
                      (1979).

61. Sinha, T.A.K. :   Re-structured Sequences of Linear
                      Positive oprators for Higher Order L_p-
                      Approximation, Ph.D. Thesis, I.I.T.
                      Kanpur (1981).

62. Stark, E.L. :   On approximation improvement for
                      trigonometric singular integrals by means
                      of finite oscillation kernels with
                      seperated zeros, in: ''Proceedings of the
                      conf. on constructive function theory'',
                      Varna (1970).

63. Stark, E.L. :   On a generalisation of Abel-Poisson's
                      singular integral having kernels of
                      finite oscillation, Studia Sci. Math.
                      Hungar, 7(1972), 437-449.

64. Stein, E. :   Singular Integrals and Differentiability
                      Properties of Functions, Princeton Univ.
                      Press, Princeton (1970).

65. Timan, A.F. :   Theory of Approximation of Functions
                      of a Real Variable, Hindustan Publishing
                      Corporation, Delhi (1966).

66. Widder, D.V. : The Laplace Transform, Princeton (1946).

67. Winslin, A. : Global L_p - Approximation by Linear Combinations of Certain Linear Positive operators, Ph.D. Thesis (to be submitted), I.I.T. Kanpur, (1982).

68. Wood, B. : Degree of L_p-approximation by certain positive convolution operators, J. Approx. Theory, 23(1978), 354-363.

69. Zygmund, A. : Trigonometric Series, Vol. I, Cambridge University Press, London-New York (1968).

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